

6.1 Laurent Series

The field of fractions of a commutative domain is the smallest field containing it. The construction of the field of fractions is modeled on the relationship between the ring of integers and the field of rational numbers.

Definition 6.1.1 (Field of fractions)**.** Fix a commutative domain A. Let $A^\times := A \setminus \{0\}$ denote the set of nonzero elements in A. Define the binary relation $\sim$ on $A \times A^\times$ by setting $(a_0, b_0) \sim (a_1, b_1)$ when $a_0 b_1 - b_0 a_1 = 0$. Since A is a commutative ring, this relation is clearly reflective and symmetric. For any $(a_0, b_0) \sim (a_1, b_1)$ and any $(a_1, b_1) \sim (a_2, b_2)$, we have

$$0 = b_2(0)+b_0(0) = b_2(a_0b_1-b_0a_1)+b_0(a_1b_2-b_1a_2) = b_1(a_0b_2-b_0a_2)\,.$$

Since $b_1 \neq 0$ and A is a domain, it follows that $a_0 b_2 - b_0 a_2 = 0$ and $(a_0, b_0) \sim (a_2, b_2)$, so this relation is also transitive. In other words, we have an equivalence relation on the product $A \times A^\times$. The set of equivalence classes is denote by $\operatorname{Frac}(A)$. Traditionally, one writes a_0/b_0 for the equivalence class of the pair (a_0, b_0).

For any two elements a_0/b_0 and a_1/b_1 in $\operatorname{Frac}(A)$, we define addition and multiplication by

$$\frac{a_0}{b_0} + \frac{a_1}{b_1} := \frac{a_0b_1 + a_1b_0}{b_0b_1} \qquad \text{and} \qquad \left(\frac{a_0}{b_0}\right)\left(\frac{a_1}{b_1}\right) := \frac{a_0a_1}{b_0b_1}\,.$$

Observe that the elements $(a_0b_1 + a_1b_0)/b_0b_1$ and a_0a_1/b_0b_1 in $\operatorname{Frac}(A)$ depend only on the equivalence classes a_0/b_0 and a_1/b_1. Indeed, if $(a_0, b_0) \sim (a_2, b_2)$ then we have $a_0b_2 = a_2b_0$ and

$$\begin{aligned}(a_0b_1 + a_1b_0)(b_2b_1) &= (a_0b_2b_1^2 + a_1b_0b_1b_2)\\ &= (a_2b_0b_1^2 + a_1b_0b_1b_2) = (a_2b_1 + a_1b_2)(b_0b_1)\,,\end{aligned}$$

$$(a_0a_1)(b_2b_1) = a_0b_2a_1b_1 = a_2b_0a_1b_1 = (a_2a_1)(b_0b_1)\,,$$

which imply that $(a_0b_1 + a_1b_0)/(b_0b_1) = (a_2b_1 + a_1b_2)/(b_2b_1)$ and $(a_0a_1)(b_0b_1) = (a_2a_1)/(b_2b_1)$. Because these definitions immediately imply that

$$\frac{0}{1} + \frac{a_0}{b_0} = \frac{a_0}{b_0} \qquad\qquad \left(\frac{1}{1}\right)\left(\frac{a_0}{b_0}\right) = \frac{a_0}{b_0}\,,$$

the additive identity is $0/1$ and the multiplicative identity is $1/1$. One easily verifies that these binary operations endow $\operatorname{Frac}(A)$ with the structure of a field. Since we also have

$$\frac{a_0}{1} + \frac{a_1}{1} = \frac{a_0 + a_1}{1} \qquad\qquad \left(\frac{a_0}{1}\right)\left(\frac{a_1}{1}\right) = \frac{a_0a_1}{1}\,,$$

the canonical map from A to $\operatorname{Frac}(A)$, which sends $a \mapsto a/1$, is compatible with addition and multiplication making $\operatorname{Frac}(A)$ into an A-algebra.

For our combinatorial applications, there are three important special cases of this general construction.

- The ***field*** $\mathbb{Q}$ ***of rational numbers*** is the field of fractions for the ring $\mathbb{Z}$ of integers.
- The ***field*** $R(x)$ ***of rational functions*** is the field of fractions for the ring $R[x]$ of polynomials in the variable x with coefficients in R.
- The ***field*** $R((x))$ ***of Laurent series*** is the field of fractions for the ring $R[[x]]$ of formal power series in the variable x with coefficients in R.

To describe a normal form for the elements in $R((x))$, we first identify the units in $R[[x]]$.

Lemma 6.1.2. *A formal power series $\sum_{j \in \mathbb{N}} a_j\, x^j \in R[[x]]$ has an inverse if and only if the constant term a_0 has an inverse in R.*

Proof. The existence of a formal power series $\sum_{j \in \mathbb{N}} b_j\, x^j$ in $R[[x]]$ satisfying

$$1 = \Big(\sum_{j \in \mathbb{N}} a_j\, x^j\Big)\Big(\sum_{j \in \mathbb{N}} b_j\, x^j\Big) = \sum_{j \in \mathbb{N}} \left(\sum_{k=0}^{j} a_k b_{j-k}\right) x^j\,,$$

is equivalent to having a solution to the system of equations:

$$a_0 b_0 = 1\,, \qquad a_0 b_1 + a_1 b_0 = 0\,, \qquad \dots\,, \qquad \sum_{k=0}^{j} a_k\, b_{j-k} = 0$$

for all positive integers j. Using this observation, we prove both directions of the claim as follows.

$\Leftarrow$: When a_0 has an inverse in R, it follows that

$$b_0 = a_0^{-1}\,, \quad b_1 = -a_0^{-1}(a_1 b_0)\,, \quad b_2 = -a_0^{-1}(a_1 b_1 + a_2 b_0)\,, \quad \cdots$$

The coefficients of the inverse are given recursively by $b_0 = a_0^{-1}$ and $b_j = -a_0^{-1} \sum_{k=1}^{j} a_k b_{j-k}$ for all positive integers j.

$\Rightarrow$: When a_0 does not have an inverse in R, the equation $a_0 b_0 = 1$ has no solutions. □

Proposition 6.1.3. *Assume that R is a field. For any nonzero Laurent series $h \in R((x))$, there exists a unique integer ℓ and a unique formal power series $\sum_{j \in \mathbb{N}} c_j\, x^j \in R[[x]]$ such that $c_0 \neq 0$ and*

$$h = x^{\ell} \left(\sum_{j \in \mathbb{N}} c_j\, x^j\right) = \sum_{j \geqslant \ell} c_{j-\ell}\, x^j\,.$$

Proof. Consider nonzero Laurent series $h \in R((x))$. By definition, there exists $f \in R[[x]]$ and a nonzero $g \in R[[x]]$ such that $h = f/g$. Setting $m := \operatorname{ord}(f)$ and $n := \operatorname{ord}(g)$, we have $f = \sum_{j \in \mathbb{N}} a_j\, x^j$ and $g = \sum_{j \in \mathbb{N}} b_j\, x^j$ where $a_m \neq 0$, $a_j = 0$ for all $0 \leqslant j < m$, $b_n \neq 0$, and $b_k = 0$ for all $0 \leqslant k < n$. The lemma establishes that the formal power series $f^* := \sum_{j\in\mathbb{N}} a_{m+j}\, x^j$ and $g^* := \sum_{j\in\mathbb{N}} b_{n+j}\, x^j$ are units in $R[[x]]$. Let $q \in R[[x]]$ be the inverse of g^*, so $q(0) \neq 0$. Hence, we obtain

$$h = \frac{f}{g} = \frac{x^m\, f^*}{x^n\, g^*} = \frac{x^m\, f^*\, q}{x^n\, g^*\, q} = x^{n-m}(f^*\, q)\,,$$

so setting $\ell := n - m$ and $\sum_{j\in\mathbb{N}} c_j\, x^j := f^*\, q$ proves existence.

To see uniqueness, suppose that we have

$$h = x^{\ell}\left(\sum_{j\in\mathbb{N}} c_j\, x^j\right) = x^{\ell'}\left(\sum_{j\in\mathbb{N}} c'_j\, x^j\right)$$

where $c_0 \neq 0$ and $c'_0 \neq 0$. Choosing $k >\!\!\geqslant \max(|\ell|, |\ell'|)$, we see that $x^{k+\ell} \sum_{j\in\mathbb{N}} c_j\, x^j = x^{k+\ell'} \sum_{j\in\mathbb{N}} c'_j\, x^j$ are nonzero elements in $R[[x]]$. Comparison of orders shows that $k + \ell = k + \ell'$ which means $\ell = \ell'$. Finally, dividing by x^{ℓ}, which is a unit in $R((x))$, we conclude that $\sum_{j\in\mathbb{N}} c_j\, x^j = \sum_{j\in\mathbb{N}} c'_j\, x^j$. □

Remark 6.1.4. The polynomial ring $R[x]$ is canonical embedded into algebra of formal power series $R[[x]]$ defined by sending the polynomial $a_0 + a_1\, x + \cdots + a_n\, x^n \in R[x]$ to the formal power series $a_0 + a_1\, x + \cdots + a_n\, x^n + \sum_{j>n} 0\, x^j \in R[[x]]$. This map induces a canonical embedding of $R(x)$ into $R((x))$, so we may regard any rational function as a Laurent series.

Problem 6.1.5 (Geometric series)**.** Show that the rational function $(1-x)^{-1}$ in $\mathbb{Z}(x)$ equals the formal power series $\sum_{j\in\mathbb{N}} x^j$ in $\mathbb{Z}((x))$.

Solution. We have

$$(1-x)\Big(\sum_{j\in\mathbb{N}} x^j\Big) = \sum_{j\in\mathbb{N}} x^j - \sum_{j\geqslant 1} x^j = 1 + \sum_{j\geqslant 1} x^j - \sum_{j\geqslant 1} x^j = 1\,. \qquad \square$$

Exercises

Problem 6.1.6. Let K be a field of characteristic zero and consider the K-algebra $K((x))$ of formal Laurent series. The ***formal residue*** map $\operatorname{Res}\colon K((x)) \to K$ is defined by $\operatorname{Res}(f) := [x^{-1}](f)$. For any two $f, g \in K((x))$, prove the following:

(i) $\operatorname{Res}\big(\frac{df}{dx}\big) = 0$; *Hint*: Differentiation is defined term-by-term.

(ii) $\operatorname{Res}\big(\frac{df}{dx}\, g\big) = -\operatorname{Res}\big(f\, \frac{dg}{dx}\big)$; *Hint*: Assume the product rule holds.

(iii) $\operatorname{Res}\big(\frac{1}{f}\, \frac{df}{dx}\big) = \operatorname{ord}(f)$ for all $f \neq 0$.

6.2 Formal Derivatives

A surprising amount of calculus extends to formal power series.

Definition 6.2.1. For any formal power series $f := \sum_{j\in\mathbb{N}} a_j\, x^j$ in $R[[x]]$, its ***derivative*** is $\frac{df}{dx} := \sum_{j\in\mathbb{N}} (j+1)\, a_{j+1}\, x^j \in R[[x]]$.

This operation satisfies the usual rules.

Proposition 6.2.2 (Differentiation rules)**.** *For all $f, g \in R[[x]]$ and all $r, s \in R$, we have the following:*

$$\text{(Linearity)} \qquad \frac{d}{dx}(r\, f + s\, g) = r\, \frac{df}{dx} + s\, \frac{dg}{dx}\,,$$

$$\text{(Product rule)} \qquad \frac{d}{dx}(f\, g) = \frac{df}{dx}\, g + f\, \frac{dg}{dx}\,,$$

(Kernel) *The equation* $\dfrac{df}{dx} = 0$ *implies that* $f = f(0) \in R$.

Proof. When $f := \sum_{j \in \mathbb{N}} a_j\, x^j$ and $g := \sum_{j \in \mathbb{N}} b_j\, x^j$, we have

$$\begin{aligned}\frac{d}{dx}(r\, f + s\, g) &= \frac{d}{dx}\left[\sum_{j \in \mathbb{N}} (r\, a_j + s\, b_j)\, x^j\right] = \sum_{j \in \mathbb{N}} (j+1)(r\, a_{j+1} + s\, b_{j+1})\, x^j \\ &= r\left[\sum_{j \in \mathbb{N}} (j+1)\, a_{j+1}\, x^j\right] + s\left[\sum_{j \in \mathbb{N}} (j+1)\, b_{j+1}\, x^j\right] = r\, f' + s\, g'\,.\end{aligned}$$

and

$$\begin{aligned}\frac{d}{dx}(f\, g) &= \frac{d}{dx}\left[\sum_{j \in \mathbb{N}}\left(\sum_{k=0}^{j} a_k\, b_{j-k}\right) x^j\right] = \sum_{j \in \mathbb{N}}\left((j+1)\sum_{k=0}^{j+1} a_k\, b_{j+1-k}\right) x^j \\ &= \sum_{j \in \mathbb{N}}\left(\sum_{k=0}^{j+1} k\, a_k\, b_{j+1-k} + (j+1-k)\, a_k\, b_{j+1-k}\right) x^j \\ &= \sum_{j \in \mathbb{N}}\left[\left(\sum_{k=0}^{j} (k+1)\, a_{k+1}\, b_{j-k}\right) + \left(\sum_{k=0}^{j} (j+1-k)\, a_k\, b_{j+1-k}\right)\right] x^j \\ &= \left[\sum_{j \in \mathbb{N}} (j+1)\, a_{j+1}\, x^j\right]\left[\sum_{j \in \mathbb{N}} b_j\, x^j\right] + \left[\sum_{j \in \mathbb{N}} a_j\, x^j\right]\left[\sum_{j \in \mathbb{N}} (j+1)\, b_{j+1}\, x^j\right] \\ &= \frac{df}{dx}\, g + f\, \frac{dg}{dx}\,.\end{aligned}$$

The equation $df/dx = 0$ implies that, for all $j \in \mathbb{N}$, we have $(j+1)\, a_{j+1} = 0$. Since the coefficient ring R is a domain having characteristic zero, the number $j+1$ is a nonzerodivisor in R. Therefore, we deduce that $a_{j+1} = 0$ for all $j \in \mathbb{N}$, so $f = a_0$. □

Having built a rigorous foundation, we return to the variant of the binomial theorem [2.3.5] for multichoose coefficients.

Theorem 6.2.3 (Generalized binomial)**.** *For any nonnegative integer n, we have*

$$\frac{1}{(1-x)^n} = \sum_{j \in \mathbb{N}} \left(\!\!\binom{n}{j}\!\!\right) x^j = \sum_{j \in \mathbb{N}} \binom{n+j-1}{j} x^j\,.$$

Inductive proof. When $n = 0$, we have $1 = \sum_{j \in \mathbb{N}} \left(\!\binom{0}{j}\!\right) x^j$, so the base case holds. Suppose that given identity holds from some nonnegative integer n. Differentiating the induction hypothesis and using the absorption identity [2.3.1] for multichoose coefficients gives

$$\begin{aligned}
& & \frac{d}{dx}\left(\frac{1}{(1-x)^n}\right) &= \frac{d}{dx}\left[\sum_{j\in\mathbb{N}}\left(\!\!\binom{n}{j}\!\!\right)x^j\right] \\
&\Leftrightarrow & \frac{n}{(1-x)^{n+1}} &= \sum_{j\in\mathbb{N}}(j+1)\left(\!\!\binom{n}{j+1}\!\!\right)x^j \\
&\Leftrightarrow & \frac{n}{(1-x)^{n+1}} &= \sum_{j\in\mathbb{N}}n\left(\!\!\binom{n+1}{j}\!\!\right)x^j \\
&\Leftrightarrow & \frac{1}{(1-x)^{n+1}} &= \sum_{j\in\mathbb{N}}\left(\!\!\binom{n+1}{j}\!\!\right)x^j = \sum_{j\in\mathbb{N}}\binom{n+j}{j}x^j\,. \qquad \square
\end{aligned}$$

Differentiation allows one to extract coefficients.

Proposition 6.2.4 (Maclaurin series)**.** *Assume that the coefficient ring R contains $\mathbb{Q}$. For any formal power series $f \in R[[x]]$, we have*

$$f = \sum_{j\in\mathbb{N}} \frac{1}{j!}\left(\left.\frac{d^j f}{dx^j}\right|_{x=0}\right)x^j\,.$$

Proof. Consider the formal power series $f := \sum_{j\in\mathbb{N}} a_j\,x^j \in R[[x]]$. For any nonnegative integer k, repeated differentiation gives

$$\frac{d^k f}{dx^k} = \sum_{j\in\mathbb{N}}(j+1)(j+2)\cdots(j+k)\,a_{j+k}\,x^j\,,$$

so we deduce that $\left.\frac{d^k f}{dx^k}\right|_{x=0} = k!\,a_k$. Since R contains $\mathbb{Q}$, we may divide by $k!$. $\square$

As an application, we extend the binomial theorem to any complex exponent.

Problem 6.2.5. For any complex number r, show that

$$(1+x)^r = \sum_{j\in\mathbb{N}}\binom{r}{j}x^j = \sum_{j\in\mathbb{N}}\frac{r^{\underline{j}}}{j!}\,x^j \in \mathbb{C}[[x]]\,.$$

Solution. For any nonnegative integer m, we obtain

$$\begin{aligned}
\frac{d^m}{dx^m}\left.\left(\sum_{j\in\mathbb{N}}a_j\,x^j\right)\right|_{x=0} &= \left.\left(\sum_{j\geqslant m}a_j(j)(j-1)\cdots(j-m+1)\,x^{j-m}\right)\right|_{x=0} = m!\,a_m\,, \\
\frac{d^m}{dx^m}\left.\left((1+x)^r\right)\right|_{x=0} &= r(r-1)(r-2)\cdots(r-m+1)\left.(1+x)^{r-m}\right|_{x=0} = r^{\underline{m}}\,, \\
\frac{d^m}{dx^m}\left.\left(\sum_{j\in\mathbb{N}}\frac{r^{\underline{j}}}{j!}\,x^j\right)\right|_{x=0} &= \left.\left(\sum_{j\geqslant m}\frac{r^{\underline{j}}}{j!}j(j-1)\cdots(j-m)x^{j-m}\right)\right|_{x=0} = \frac{r^{\underline{m}}}{m!}\,m! = r^{\underline{m}}\,.
\end{aligned}$$

Comparing coefficients establishes the desired equality. $\square$

Remark 6.2.6. The ring of polynomial differential operators (also known as the ***Weyl algebra***) is not commutative. For example, the product rule implies that

$$\left(\tfrac{d}{dx}x - x\tfrac{d}{dx}\right)f = \tfrac{d}{dx}\left(xf\right) - x\left(\tfrac{d}{dx}f\right) = \left(f + x\tfrac{df}{dx}\right) - x\tfrac{df}{dx} = f\,,$$

so $\frac{d}{dx}x - x\frac{d}{dx} = 1$. As a consequence, we have

$$\left(x\tfrac{d}{dx}\right)^2 = \left(x\tfrac{d}{dx}\right)\left(x\tfrac{d}{dx}\right) = x\left(\tfrac{d}{dx}x\right)\tfrac{d}{dx} = x\left(x\tfrac{d}{dx}+1\right)\tfrac{d}{dx} = x^2\tfrac{d^2}{dx^2} + x\tfrac{d}{dx}\,.$$

Problem 6.2.7. Let $\partial := \frac{d}{dx}$ denote the basic differential operator on $\mathbb{Q}[[x]]$. For any nonnegative integer n, prove that

$$(x\partial)^n = \sum_{k\in\mathbb{N}} \left\{ {n \atop k} \right\} x^k\,\partial^k \quad\text{and}\quad x^n\partial^n = \sum_{k\in\mathbb{N}} (-1)^{n-k}\left[{n \atop k} \right](x\,\partial)^k\,.$$

Solution. For the first identity, we proceed by induction on n. When $n = 0$, we have $(x\partial)^0 = 1 = \sum_{k\in\mathbb{Z}} \left\{ {0 \atop k} \right\} x^k\,\partial^k$, so the base case holds. Assume that the formula holds for some nonnegative integer n. The induction hypothesis, the product rule, reindexing the sum, and the addition formula [3.0.3] for Stirling subset numbers give

$$\begin{aligned}(x\partial)^{n+1} &= (x\partial)(x\partial)^n = x\partial\left(\sum_{k\in\mathbb{N}} \left\{ {n \atop k} \right\} x^k\,\partial^k\right)\\ &= \sum_{k\in\mathbb{Z}} k\left\{ {n \atop k} \right\} x^k\,\partial^k + \sum_{k\in\mathbb{Z}} \left\{ {n \atop k} \right\} x^{k+1}\,\partial^{k+1}\\ &= \sum_{k\in\mathbb{Z}} \left(k\left\{ {n \atop k} \right\} + \left\{ {n \atop k-1} \right\}\right) x^k\,\partial^k = \sum_{k\in\mathbb{Z}} \left\{ {n+1 \atop k} \right\} x^k\,\partial^k\,.\end{aligned}$$

For any two nonnegative integers m and n, Stirling inversion [5.2.4] asserts that $\sum_{k\in\mathbb{Z}}(-1)^{n-k}\left[{n \atop k} \right]\left\{ {k \atop m} \right\} = \delta_{m,n}$. Hence, the first identity implies that

$$\begin{aligned}\sum_{k\in\mathbb{Z}} (-1)^{n-k}\left[{n \atop k} \right](x\,\partial)^k &= \sum_{k\in\mathbb{Z}} (-1)^{n-k}\left[{n \atop k} \right]\left(\sum_{m\in\mathbb{Z}} \left\{ {k \atop m} \right\} x^m\,\partial^m\right)\\ &= \sum_{m\in\mathbb{Z}} \left(\sum_{k\in\mathbb{Z}} (-1)^{n-k}\left[{n \atop k} \right]\left\{ {k \atop m} \right\}\right) x^m\,\partial^m\\ &= \sum_{m\in\mathbb{Z}} \delta_{m,n}\, x^m\,\partial^m = x^n\,\partial^n\,.\end{aligned}$$

□

Exercises

Problem 6.2.8. The ***exponential power series*** is defined to be

$$\exp(x) := \sum_{n\in\mathbb{N}} \frac{x^n}{n!} \in \mathbb{Q}[[x]]\,.$$

(i) Let $f \in \mathbb{Q}[[x]]$. If $\frac{df}{dx} = f$, then show that there exists $c \in \mathbb{Q}$ such that $f = c\,\exp(x)$.

(ii) By extracting coefficients, show that the binomial theorem is equivalent to the identity

$$\exp\big(t(x+y)\big) = \exp(tx)\exp(ty) \in \mathbb{Q}[[t,x,y]]\,.$$

(iii) For all nonnegative integers k and n, use a similar approach to prove the multinomial theorem

$$(x_1 + x_2 + \cdots + x_k)^n = \sum_{j_1+j_2+\cdots+j_k=n} \frac{n!}{j_1!\,j_2!\cdots j_k!} x_1^{j_1} x_2^{j_2}\cdots x_k^{j_k}\,.$$

6.3 Coefficient Extraction

The operators that isolate individual coefficients from a formal power series provide the crucial tools for our applications.

Definition 6.3.1. For any nonnegative integer m, the ***coefficient extraction*** function $[x^m] : R[[x]] \to R$ is defined by

$$[x^m]\Big(\sum_{j \in \mathbb{N}} a_j \, x^j\Big) = a_m \, .$$

The definition of addition and multiplication in $R[[x]]$ establish that this are R-linear operators.

Problem 6.3.2. For all nonnegative integers m and n, use the equation $(1{+}x)^m \, (1{+}x)^n = (1{+}x)^{m+n}$ to reprove the Vandermonde identity [2.1.5] for binomial coefficients.

Solution. The binomial theorem [2.1.6] is equivalent to having $[x^k]\big((1+x)^n\big) = \binom{n}{k}$ for all integers k. Hence, the definition for multiplication of formal power series gives

$$\begin{aligned}\sum_{j \in \mathbb{Z}} \binom{m}{j}\binom{n}{k-j} &= \sum_{j=0}^{k} \binom{m}{j}\binom{n}{k-j} = [x^k]\big((1+x)^m(1+x)^n\big) \\ &= [x^k]\big((1+x)^{m+n}\big) = \binom{m+n}{k}. \end{aligned} \qquad \square$$

Problem 6.3.3. For any formal power series $f := \sum_{j \in \mathbb{N}} a_j \, x^j$ in $R[[x]]$ and any nonnegative integer m, find $[x^m]\big((1-x)^{-1} f\big)$.

Solution. The geometric series [6.1.5] satisfies $(1-x)^{-1} = \sum_{j \in \mathbb{N}} x^j$. Hence, the definition for multiplication in $R[[x]]$ gives

$$[x^m]\left[\Big(\sum_{j \in \mathbb{N}} x^j\Big) f\right] = [x^m]\left[\sum_{j \in \mathbb{N}} \Big(\sum_{k=0}^{j} a_k\Big) x^j\right] = \sum_{k=0}^{m} a_k \, . \qquad \square$$

Remark 6.3.4. When $f := (1-x)^{-1} = \sum_{j \in \mathbb{N}} x^j$, the previous problem shows

$$\frac{1}{(1-x)^2} = \sum_{j \in \mathbb{N}} (j+1)\, x^j \qquad \text{and} \qquad \frac{x}{(1-x)^2} = \sum_{j \in \mathbb{N}} j \, x^j \, .$$

Proposition 6.3.5. *For any polynomial p in $R[x]$ and any formal power series $f := \sum_{j \in \mathbb{N}} a_j \, x^j$ in $R[[x]]$, we have*

$$p\big(x \tfrac{d}{dx}\big) f = \sum_{j \in \mathbb{N}} p(j) \, a_j \, x^j$$

Inductive proof. For all nonnegative integers m, it suffices to prove that $\big(x \frac{d}{dx}\big)^m f = \sum_{j \in \mathbb{N}} j^m \, a_j \, x^j$. When $m = 0$, we have $\big(x \frac{d}{dx}\big)^0 f = f = \sum_{j \in \mathbb{N}} a_j \, x^j = \sum_{j \in \mathbb{N}} j^0 \, a_j \, x^j$, so the base case holds.

Assume that the claim holds for some nonnegative integer m. The induction hypothesis and properties of the derivative [6.2.2] yield

$$\begin{aligned}\left(x\tfrac{d}{dx}\right)^{m+1} f &= \left(x\tfrac{d}{dx}\right)\left(x\tfrac{d}{dx}\right)^{m} f = x\frac{d}{dx}\left(\sum_{j\in\mathbb{N}} j^m\, a_j\, x^j\right) = x\left(\sum_{j\in\mathbb{N}} (j+1)^m\, a_{j+1}\, x^j\right)\\ &= \sum_{j\in\mathbb{N}} (j+1)^m\, a_{j+1}\, x^{j+1} = \sum_{j\in\mathbb{N}} j^m\, a_j\, x^j\,. \qquad \square\end{aligned}$$

Problem 6.3.6. Use the previous proposition to reprove the Absorption Identity [2.1.3] for binomial coefficients.

Solution. Applying proposition for the polynomial $x \in \mathbb{Z}[x]$ gives

$$\begin{aligned}\sum_{k\in\mathbb{N}} k\binom{n+1}{k} x^k &= \left(x\frac{d}{dx}\right)\left((1+x)^{n+1}\right) = (n+1)\,x\,(1+x)^n\\ &= \sum_{k\in\mathbb{N}} (n+1)\binom{n}{k} x^{k+1} = \sum_{k\in\mathbb{N}} (n+1)\binom{n}{k-1} x^k\,,\end{aligned}$$

so extracting the coefficient of x^k completes the proof. $\square$

Problem 6.3.7. Prove that $\log\left(\dfrac{1}{1-x}\right) = \displaystyle\sum_{j\in\mathbb{N}} \frac{x^{j+1}}{j+1}$ in $\mathbb{Q}[[x]]$.

Solution. Suppose that $\log(\frac{1}{1-x}) = \sum_{j\in\mathbb{N}} a_j\, x^j$ in $\mathbb{Q}[[x]]$. The proposition and the geometric series [6.1.5] give

$$\begin{aligned}\sum_{j\in\mathbb{N}} j\, a_j\, x^j &= \left(x\frac{d}{dx}\right)\log\left(\frac{1}{1-x}\right)\\ &= x\left(\frac{1}{(1-x)^{-1}}\right)\left(\frac{-1}{(1-x)^2}\right)(-1) = \frac{x}{1-x} = \sum_{j>0} x^j\,.\end{aligned}$$

Comparing coefficients establishes that $j\, a_j = 1$ for all positive integers j. $\square$

Problem 6.3.8. Find a closed formula for the formal power series $\sum_{j\in\mathbb{N}}(j^2+4j+5)\,\frac{x^j}{j!}$.

Solution. Applying the proposition, we have

$$\begin{aligned}&\textstyle\sum_{j\in\mathbb{N}}(j^2+4j+5)\,\frac{x^j}{j!}\\ &= \left((x\partial)^2 + 4(x\partial) + 5\right)\exp(x)\\ &= (x\partial)\left(x\exp(x)\right) + 4x\exp(x) + 4\exp(x)\\ &= x\left(\exp(x) + x\exp(x)\right) + 4x\exp(x) + 4\exp(x)\\ &= (x^2+5x+5)\exp(x)\,. \qquad \square\end{aligned}$$

Exercises

Problem 6.3.9. Find the unique sequence $(a_0, a_1, a_2, \dots)$ of real numbers such that, for all nonnegative integers j, we have

$$\sum_{k=0}^{j} a_k\, a_{j-k} = 1\,.$$