0
Two Basic Principles

We start by examining the pigeonhole principle and the principle of mathematical induction. Both of these tools are powerful, but deceptively elementary. One may need real inspiration to apply them. We illustrate the first principle with some "perfect proofs" cherished by Erdős and we explain the second with the famous Fibonacci numbers.

Notation 0.0.1. Throughout, $\mathbb{N} := \{0, 1, 2, 3, \dots\}$ denotes the set of nonnegative integers. To ensure that the cardinality of every finite set appears in this set, it must include zero.

0.0 The Pigeonhole Principle

Our first principle builds on the simple observation that there are no injective mappings from a finite set into a small set. Despite being seemingly obvious, this statment does nevertheless produce unexpected results.

Theorem 0.0.2 (Pigeonhole Principle)**.** *Let n and r be nonnegative integers. If n objects are placed in r boxes where $r < n$, then at least one of the boxes contains more than one object.*

Proof by contradiction. Assume that no box has more than one object. If there is no box with at least two objects, then each of the r boxes has either zero or one object in it. Let m be the number of boxes with zero objects in them, so $m \geqslant 0$. It follows that there are $r - m$ boxes with one object. However, this means that the total number of objects placed in the r boxes is $r - m \leqslant r < n$ which is a contradiction. Therefore, our assumption is false and there exists at least one box with more than one object. □

Problem 0.0.3. When you pick 5 cards from a standard 52-card deck of French playing cards (the most common deck used today), demonstrate that at least 2 will have the same suit.

French, German, Italian, and Swiss decks of playing cards all have the same number of suits. The four suits in a French deck are clubs (♣), diamonds (♢), hearts (♡), and spades (♠).

Solution. Since there are 5 cards (the objects) and 4 suits (the boxes), the Pigeonhole Principle 0.0.2 establishes that at least 2 cards have the same suit. □

Problem 0.0.4. Given 10 points in the unit square, show that there exists 2 points that are at most $\sqrt{2}/3$ units apart.

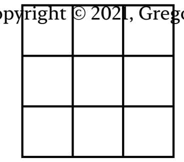

Figure 0.1: Subdivided unit square

Solution. Subdivide the unit square into 9 smaller squares all having the same area. Since there are 10 points and 9 smaller squares, the Pigeonhole Principle 0.0.2 implies that 2 points lie in the same smaller square. In a square, the points on the opposite corners have greatest distance between them. In our small squares, the Pythagorean theorem implies that the distance is at most $\sqrt{(1/3)^2 + (1/3)^2} = \sqrt{2}/3$. □

Problem 0.0.5. Let n be a positive integer. Given n integers $a_1, a_2, \dots, a_n$, prove that there is a subset of consecutive numbers $a_{j+1}, a_{j+2}, \dots, a_k$ with $j < k$ such that sum $a_{j+1} + a_{j+2} + \cdots + a_k$ is a multiple of n.

Solution. Fix two sets: $\mathcal{N} := \{0, a_1, a_1 + a_2, \dots, a_1 + a_2 + \cdots + a_n\}$ and $\mathcal{R} := \{0, 1, \dots, n-1\}$. Consider the map $f : \mathcal{N} \to \mathcal{R}$ where $f(m)$ is the remainder of m upon divison by n. Since the cardinalities of our two sets satisfy the inequality $|\mathcal{N}| = n + 1 > n = |\mathcal{R}|$, the Pigeonhole Principle 0.0.2 implies that there are two sums $a_1 + a_2 + \cdots + a_j$ and $a_1 + a_2 + \cdots + a_k$, for some $0 \leqslant j < k \leqslant n$, with same remainder. It follows that

$$a_{j+1} + a_{j+2} + \cdots + a_k = (a_1 + a_2 + \cdots + a_j) - (a_1 + a_2 + \cdots + a_k)$$

is a multiple of n. □

Lejeune Dirichlet (1805–1859) made deep contributions to number theory, Fourier series, and other topics in analysis.

Dirichlet first formulated the Pigeonhole Principle to establish good rational approximations for irrational numbers.

Theorem 0.0.6 (Dirichlet 1879)**.** *For any real number x and any positive integer n, there exists a rational number p/q such that $p, q \in \mathbb{Z}$, $1 \leqslant q \leqslant n$, and*

$$\left| x - \frac{p}{q} \right| < \frac{1}{nq} \leqslant \frac{1}{q^2}\,.$$

The Roth Theorem shows that this is essentially the tightest possible. The bound on rational approximation of algebraic numbers cannot be improved by increasing the exponent beyond 2.

Proof. Let $\operatorname{frac}(x) := x - \lfloor x \rfloor$ denote the ***fractional part*** of the real number x, so $0 \leqslant \operatorname{frac}(x) < 1$. Consider the $n+1$ real numbers $\operatorname{frac}(jx)$, where j is a positive integer such that $1 \leqslant j \leqslant n+1$, and the n half-open intervals $[0, 1/n), [1/n, 2/n), \dots, [(n-1)/n, n/n)$ that subdivide the unit interval $[0, 1)$. The Pigeonhole Principle 0.0.2 implies that there exists a interval containing more than one of the numbers. If the numbers $\operatorname{frac}(jx)$ and $\operatorname{frac}(kx)$, where $j < k$, belong to the same interval, then their difference is less than $1/n$. Setting $q := k - j$ and $p := \lfloor kx \rfloor - \lfloor jx \rfloor$, we have

$$\begin{aligned} \frac{1}{n} &> |\operatorname{frac}(kx) - \operatorname{frac}(jx)| = |kx - \lfloor kx \rfloor - jx + \lfloor jx \rfloor| \\ &= |(k-j)x - (\lfloor kx \rfloor - \lfloor jx \rfloor)| = |qx - p| \end{aligned}$$

which yields $|x - p/q| < 1/nq$. □

Exercises

Problem 0.0.7. Show that at least 2 Queen's students have the same number of eyelashes.

Problem 0.0.8. Hartsfield–Jackson Atlanta International Airport (ATL) is world's busiest airport as measured by annual aircraft movements. One movement is a landing or takeoff of an aircraft. In 2015, ATL had 882 497 movements. Show that two movements occurred within a minute of each other.

Problem 0.0.9. Prove that every odd integer has a multiple that is one less than a power of two.

Problem 0.0.10. Given 5 points in a unit equilateral triangle, show that there exists two points that are at most $1/\sqrt{3}$ units apart.

Problem 0.0.11. Let m denote the arithmetic mean of a finite set of real numbers. Use the Pigeonhole Principle 0.0.2 to show that there exists at least one number in the collection that is less than or equal to m.

Problem 0.0.12. The ***degree*** of a vertex in a graph is the number of edges adjacent to it. Given a graph G with n vertices such that every vertex has degree at least $(n-1)/2$, show that G is connected.

0.1 Erdős' Favourites

Paul Erdős spoke of 'The Book' in which God records the most beautiful proofs and even asserted "You don't have to believe in God, but you should believe in The Book". We examine several arguments that Erdős personally designated as from The Book.

Paul Erdős (1913–1996) was one of the most prolific mathematicians. He is known for his social practice of mathematics (he had more than 500 collaborators) and his eccentric lifestyle.

Notation 0.1.1. For all nonnegative integers n, set

$$[n] := \{1, 2, 3, \dots, n\},$$

so $[0] = \varnothing$, $[1] = \{1\}$, $[2] = \{1, 2\}$, and $[3] = \{1, 2, 3\}$.

Proposition 0.1.2. *Let n be a positive integer. Any subset of the set $[2n]$ having cardinality $n+1$ contains two relatively prime integers.*

Proof. Partition the set $[2n]$ into the n disjoint subsets:

$$\{1, 2\}, \{3, 4\}, \{5, 6\}, \dots, \{2n-1, 2n\}.$$

View the given subset of $[2n]$ as placing $n+1$ objects into these boxes. The Pigeonhole Principle 0.0.2 implies that there exists box with at least two objects. Hence, the given subset contains two consecutive integers. Since 2 positive integers that differ by 1 are relatively prime, any subset of $[2n]$ having cardinality $n+1$ contains a pair of relatively prime integers. □

Proposition 0.1.3. *Let n be a positive integer. Any subset of the set $[2n]$ having cardinality $n+1$ contains a pair of integers such that one divides the other.*

Proof. Write every number in the given subset in the form $2^k m$ where m is an odd number between 1 and $2n-1$. Since there are $n+1$ numbers in the subset, but only n different odd parts, the Pigeonhole Principle 0.0.2 implies that there are two numbers with the same odd part. Thus, one is a multiple of the other. □

Theorem 0.1.4 (Generalized Pigeonhole Principle)**.** *Let n and r be positive integers such that $n > r$. When n objects are placed in r boxes, there exists a box containing at least $\lceil n/r \rceil$ objects.*

Proof. In terms of a function, this principle asserts that, for any two finite sets $\mathcal{N}$ and $\mathcal{R}$ satisfying $|\mathcal{N}| = n > r = |\mathcal{R}|$ and any function $f : \mathcal{N} \to \mathcal{R}$, there exists an element $m \in \mathcal{R}$ such that the preimage has cardinality at least n/r: $|f^{-1}(m)| \geqslant \lceil n/r \rceil$. Otherwise, we would have $|f^{-1}(m)| < n/r$, for all $m \in \mathcal{R}$, and $n = \sum_{m \in \mathcal{R}} |f^{-1}(m)| < r\,(n/r) = n$ which is absurd. □

Problem 0.1.5. Show that there are at least 5 people currently living in the Greater Toronto Area who were born in the same hour of the same day of the same year.

Solution. Since the verified oldest person lived less that 123 years, we may assume that all the residents of the Greater Toronto Area are at most 125 years old. Each year has at most 366 days and each day has 24 hours, so we have a total of $(125)(366)(24) = 1\,098\,000$ boxes. There are at least $6\,417\,516$ Torontonians (as counted by the 2016 Canadian Census). Thus, the Generalized Pigeonhole Principle 0.1.4 implies that at least $6417516 \big/ 1098000 \approx 5.84 > 5$ people were born within 60 minutes of each other. □

Jeanne Calment (1875–1997) lived 122 years and 164 days.

The next application is more sophisticated.

Theorem 0.1.6 (Erdős–Szekeres 1939)**.** *Let m and n be nonnegative integers. Given a finite sequence $a_1, a_2, \dots, a_{mn+1}$ of $mn+1$ distinct real numbers, there exists an increasing subsequence of length $m+1$ or a deceasing subsequence of length $n+1$.*

Proof. We demonstrate that there exists indices $i_1 < i_2 < \cdots < i_{m+1}$ such that $a_{i_1} < a_{i_2} < \cdots < a_{i_{m+1}}$ or $j_1 < j_2 < \cdots < j_{n+1}$ such that $a_{j_1} > a_{j_2} > \cdots > a_{j_{n+1}}$. Associate to each term a_i the length ℓ_i of a longest increasing subsequence starting with a_i. If $\ell_i \geqslant m+1$ for some i, then there would be an increasing subsequence of length $m+1$. Otherwise, we would have $\ell_i \leqslant m$ for all $1 \leqslant i \leqslant mn+1$. For the function $a_i \mapsto \ell_i$ mapping the set $\{a_1, a_2, \dots, a_{mn+1}\}$ into the set $[m] = \{1, 2, \dots, m\}$, the Generalized Pigeonhole Principle 0.1.4

implies that there is an element $r \in [m]$ such that the preimage has cardinality at least

$$\left\lceil \frac{mn+1}{m} \right\rceil = \left\lceil n + \frac{1}{m} \right\rceil = n+1\,.$$

Let $a_{j_1}, a_{j_2}, \dots, a_{j_{n+1}}$, where $j_1 < j_2 < \cdots < j_{n+1}$, be the terms in preimage of such an element r. Look at two consecutive terms $a_{j_k}, a_{j_{k+1}}$ for some $1 \leqslant k \leqslant n$. If $a_{j_k} < a_{j_{k+1}}$, then we would have an increasing subsequence of length r starting at $a_{j_{k+1}}$ and, consequently, an increasing subsequence of length $r+1$ starting at a_{j_k} which is impossible as $\ell_{j_k} = r$. Thus, we obtain a decreasing subsequence $a_{j_1} > a_{j_2} > \cdots > a_{j_{n+1}}$ of length $n+1$. □

Exercises

Problem 0.1.7. Fix an integer n that is greater than 1 and select n different integers from the set $[2n] := \{1, 2, 3, \dots, 2n\}$. Is it true that, among the selected integers, there will be two that are relatively prime? Is it true that, among the selected integers, there will be two such that one divides the other?

Problem 0.1.8. Fix an integer n that is greater than 1 and select $n+1$ different integers from the set $[2n] := \{1, 2, 3, \dots, 2n\}$. Is it true that, among the selected integers, there will be two such that one is equal to twice the other?

Problem 0.1.9. Given 5 integers between 1 and 8, show that 2 of them must add up to 9.

Problem 0.1.10. Given 6 people, show that there are either 3 mutual acquaintances or 3 mutual strangers.

Problem 0.1.11. Suppose that each point in real plane with integer coordinates between 0 and 99 are coloured red, yellow, or blue. Prove that there exists a rectangle whose vertices all have the same colour.

Problem 0.1.12. Suppose a standard 8×8 chessboard has two diagonally opposite corners removed, leaving 62 squares. Is it possible to place 31 dominoes (which cover squares that share a common edge) so as to cover all of these squares?

Problem 0.1.13. Demonstrate that lossless data compression algorithms cannot guarantee compression for all input.

0.2 Mathematical Induction

Our second principle is part of the foundations of arithmetic and is usually stated as an axiom of the set of nonnegative integers. To be more precise, a ***property of the set*** $\mathbb{N}$ is defined to be a function $P \colon \mathbb{N} \to \{\textsf{true}, \textsf{false}\}$. The idea is that $P(n)$ holds if and only if

$P(n) = \textsf{true}$; otherwise we have $P(n) = \textsf{false}$. The most common form of induction is prescribed as follows.

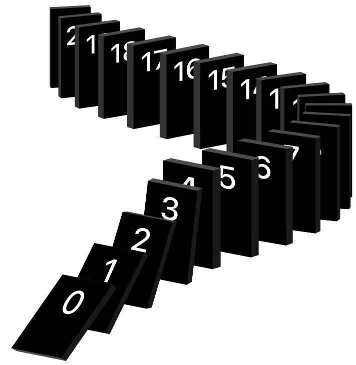

Figure 0.2: Induction is like toppling dominoes

Axiom 0.2.1 (Induction)**.** To verify that a property $P(n)$ holds for all $n \in \mathbb{N}$, it is enough to prove

Base case: $P(0)$ holds, and

Induction step: for all $n \in \mathbb{N}$, the assumption that $P(n)$ holds implies that the property $P(n+1)$ holds.

The next two problems typify the basic use of induction.

Problem 0.2.2. For all positive integers n, verify that

$$\sum_{j=0}^{n-1} (2j+1) = n^2 \,.$$

Inductive solution. When $n = 1$, we have $2(0)+1 = (1)^2$, so the base case holds. Assuming that $\sum_{j=0}^{n-2} (2j+1) = (n-1)^2$ holds, we show that the equation $\sum_{j=0}^{n-1} (2j+1) = n^2$ also holds. The induction step is

$$\begin{aligned} \sum_{j=0}^{n-1} (2j+1) &= \left[\sum_{j=0}^{n-2} (2j+1) \right] + \Big[2(n-1)+1 \Big] \\ &= (n-1)^2 + 2(n-1) + 1 = \big((n-1)+1 \big)^2 = n^2 \,. \qquad \square \end{aligned}$$

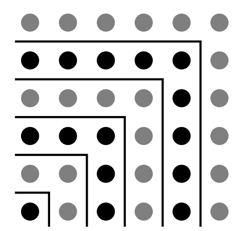

Figure 0.3: Sum of odd integers

Remark 0.2.3. Despite establishing the correctness of the formula, the induction solution to Problem 0.2.2 is unsatisfying. It feels overly formal and does not seem to explain the true origins of this equation. Figure 0.3 suggests a better way to understand this sum.

Problem 0.2.4. For all integers n greater than or equal to 4, prove that $2^n \geqslant n^2$.

Inductive solution. For all $n \geqslant 2$, we first prove, by induction, that $n^2 \geqslant 2$. When $n = 2$, we have $2^2 = 4 > 2$, so the base case holds. Assuming that the inequality $n^2 - 2 \geqslant 0$ holds, we show that $(n+1)^2 - 2 \geqslant 0$ also holds. The induction step is

$$(n+1)^2 - 2 = (n^2 + 2n + 1) - 2 \geqslant 2 + 2n - 1 = 2n + 1 \geqslant 5 \,.$$

For all $n \geqslant 4$, we now prove, via induction on n, that $2^n \geqslant n^2$. When $n = 4$, we have $4^2 = 16 = 2^4$, so the base case holds. Assuming that the inequality $2^n - n^2 \geqslant 0$ holds, we show that $2^{n+1} - (n+1)^2 \geqslant 0$ also holds. For the induction step, we have

$$\begin{aligned} 2^{n+1} - (n+1)^2 &= 2\big(2^n\big) - n^2 - 2\,n - 1 \\ &\geqslant 2\big(n^2\big) - n^2 - 2\,n - 1 = (n-1)^2 - 2 \,. \end{aligned}$$

Since the first paragraph establishes that $(n-1)^2 - 2 \geqslant 0$, we deduce that $2^{n+1} - (n+1)^2 \geqslant 0$ as required. $\square$

Among the reformulations of induction, we single out two.

Theorem 0.2.5 (Well-ordering of $\mathbb{N}$)**.** *Every nonempty subset of the set $\mathbb{N}$ of nonnegative integers has a least element.*

Proof by contradiction. Let $\mathcal{A} \subseteq \mathbb{N}$ be any nonempty subset. Suppose that $\mathcal{A}$ does not have a least element. For all $n \in \mathbb{N}$, let $P(n)$ be the property that no element of $\mathbb{N}$ strictly smaller than n lies in $\mathcal{A}$. We verify, by induction on n, that $P(n)$ holds. However, the fact that the property $P(n)$ holds for all $n \in \mathbb{N}$ implies that $\mathcal{A} = \varnothing$ which is a contradiction.

The base case $P(0)$ holds vacuously. Assuming that $P(n)$ holds, we show that the property $P(n+1)$ also holds. For any $m < n+1$, we have either $m < n$ or $m = n$. We treat these cases separately.

$m < n$: The induction hypothesis asserts that $P(n)$ holds. It follows that $m \notin \mathcal{A}$.

$m = n$: If n were to belong to the set $\mathcal{A}$, then the induction hypothesis would imply that no $k < n$ would belong to $\mathcal{A}$, so n would be the least element of $\mathcal{A}$. Since this contracts the supposition that the set $\mathcal{A}$ contains no least element, we deduce that $n \notin \mathcal{A}$.

Therefore, we conclude that $m < n+1$ implies that $m \notin \mathcal{A}$ which means that $P(n+1)$ holds. □

The next variant is especially useful when multiple instances of the inductive hypothesis are required for each inductive step.

Theorem 0.2.6 (Complete induction)**.** *To verify that a property $P(n)$ holds for all nonnegative integers n, it is enough to prove that*

Base case: *$P(0)$ holds, and*

Induction step: *for all $n \in \mathbb{N}$, the assumption that the property $P(k)$ holds for all $k \leqslant n$ implies that the property $P(n+1)$ holds.*

Proof by contradiction. Let P be a property satisfying both the base case and induction step. Suppose that $P(n)$ fails for some nonnegative integer n, so the set $\mathcal{F} := \{n \in \mathbb{N} \mid P(n) = \mathsf{false}\}$ is nonempty. Theorem 0.2.5 implies that the set $\mathcal{F}$ has a least element m, which means that $P(m) = \mathsf{false}$. Since the base case asserts that $P(0)$ holds, it follows that $m \neq 0$. As m is the least element for which $P(m) = \mathsf{false}$, we must also have $P(k) = \mathsf{true}$ for all $k < m$. However, the induction step would then imply that $P(m) = \mathsf{true}$ which is a contradiction. Therefore, the set $\mathcal{F}$ is empty and the property $P(n)$ must hold for all $n \in \mathbb{N}$. □

Exercises

Problem 0.2.7. For all nonnegative integers n, show that

$$\sum_{k=0}^{n} \frac{1}{(k+1)(k+2)} = \frac{n+1}{n+2}\,.$$

Problem 0.2.8. For all $n \in \mathbb{N}$, verify that $\sum_{j=0}^{n} j^3 = \left(\sum_{j=0}^{n} j\right)^2$.

Problem 0.2.9. For any subset $\mathcal{A} \subseteq \mathbb{N}$ such that
(a) $0 \in \mathcal{A}$, and
(b) the condition that $n \in \mathcal{A}$ implies that $n + 1 \in \mathcal{A}$,
demonstrate that $\mathcal{A} = \mathbb{N}$.

Problem 0.2.10. The ***square triangular numbers*** are defined by $N_0 := 0$, $N_1 := 1$, and $N_k := 34\,N_{k-1} - N_{k-2} + 2$ for all $k \geqslant 2$. The first few terms are $0, 1, 36, 1225, 41616, 1413721, 48024900, \dots$.
 (i) Prove that $N_{k-1}\,N_{k+1} = (N_k - 1)^2$ for all $k \geqslant 1$.
 (ii) Verify that

$$N_k = \left(\frac{(3 + 2\sqrt{2})^k - (3 - 2\sqrt{2})^k}{4\sqrt{2}}\right)^2 .$$

0.3 Fibonacci Numbers

Appearing as early as 200BC in work by Pingala on patterns in poetry, the 1202 book by Fibonacci introduces this sequence to solve a puzzle about the growth of an idealized rabbit population.

Definition 0.3.1. The ***Fibonacci numbers*** are the integers defined by $F_0 = 0$, $F_1 = 1$, and $F_n := F_{n-1} + F_{n-2}$ for all $n \geqslant 2$. The first few numbers in the sequence are $0, 1, 1, 2, 3, 5, 8, 13, 21, 34, \dots$.

This sequence has many curious properties.

Problem 0.3.2. For all $n \in \mathbb{N}$, prove $F_0 + F_1 + \cdots + F_n = F_{n+2} - 1$.

Inductive solution. When $n = 0$, we have $F_0 = 0 = 1 - 1 = F_2 - 1$, so the base case holds. Assuming that the equation

$$F_0 + F_1 + \cdots + F_n = F_{n+2} - 1$$

holds, we must show that $F_0 + F_1 + \cdots + F_{n+1} = F_{n+3} - 1$ holds. The induction hypothesis and the defining recurrence give

$$\begin{aligned} F_0 + F_1 + F_2 + \cdots + F_{n+1} &= (F_0 + F_1 + F_2 + \cdots + F_n) + F_{n+1} \\ &= (F_{n+2} - 1) + F_n \\ &= (F_{n+2} + F_{n+1}) - 1 \\ &= F_{n+3} - 1 . \end{aligned}$$ □

Problem 0.3.3. Prove that the Fibonacci number F_n is even if and only the index n is divisible by 3.

Inductive solution. Since first three terms are $F_0 = 0$, $F_1 = 1$, and $F_2 = 1$, the base cases hold. Assume that F_{3j} is even, F_{3j+1} is odd, and F_{3j+2} is odd. The next 3 Fibonacci numbers are

$$\begin{aligned} F_{3(j+1)} &= F_{3j+2} + F_{3j+1} = \text{odd} + \text{odd} = \text{even} \\ F_{3(j+1)+1} &= F_{3(j+1)} + F_{3j+2} = \text{even} + \text{odd} = \text{odd} \\ F_{3(j+1)+2} &= F_{3(j+1)+1} + F_{3(j+1)} = \text{odd} + \text{even} = \text{odd} , \end{aligned}$$

which completes the induction. □

The Fibonacci numbers have a closed-form expression that unexpectedly involves the irrational number $\sqrt{5}$.

Theorem 0.3.4 (Binet Formula)**.** *The Fibonacci numbers satisfy*

$$F_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n .$$

Although known to de Moivre in 1718, this formula is named for Jacques Binet (1786–1856).

Inductive proof. When $n = 0$ and $n = 1$, we have

$$F_0 = \tfrac{1}{\sqrt{5}}\left(\tfrac{1+\sqrt{5}}{2}\right)^0 - \tfrac{1}{\sqrt{5}}\left(\tfrac{1-\sqrt{5}}{2}\right)^0 = \tfrac{1}{\sqrt{5}} - \tfrac{1}{\sqrt{5}} = 0\,,$$
$$F_1 = \tfrac{1}{\sqrt{5}}\left(\tfrac{1+\sqrt{5}}{2}\right)^1 - \tfrac{1}{\sqrt{5}}\left(\tfrac{1-\sqrt{5}}{2}\right)^1 = \tfrac{1}{\sqrt{5}}\left(\tfrac{1+\sqrt{5}-1+\sqrt{5}}{2}\right) = 1\,,$$

so the base cases hold. The induction step is simply

$$\begin{aligned}
&F_{n-1} + F_{n-2} \\
&= \tfrac{1}{\sqrt{5}}\left(\tfrac{1+\sqrt{5}}{2}\right)^{n-1} - \tfrac{1}{\sqrt{5}}\left(\tfrac{1-\sqrt{5}}{2}\right)^{n-1} + \tfrac{1}{\sqrt{5}}\left(\tfrac{1+\sqrt{5}}{2}\right)^{n-2} - \tfrac{1}{\sqrt{5}}\left(\tfrac{1-\sqrt{5}}{2}\right)^{n-2} \\
&= \tfrac{1}{\sqrt{5}}\left(\tfrac{1+\sqrt{5}}{2}\right)^{n-2}\left(\tfrac{1+\sqrt{5}}{2} + 1\right) - \tfrac{1}{\sqrt{5}}\left(\tfrac{1-\sqrt{5}}{2}\right)^{n-2}\left(\tfrac{1-\sqrt{5}}{2} + 1\right) \\
&= \tfrac{1}{\sqrt{5}}\left(\tfrac{1+\sqrt{5}}{2}\right)^{n-2}\left(\tfrac{3+\sqrt{5}}{2}\right) - \tfrac{1}{\sqrt{5}}\left(\tfrac{1-\sqrt{5}}{2}\right)^{n-2}\left(\tfrac{3-\sqrt{5}}{2}\right) \\
&= \tfrac{1}{\sqrt{5}}\left(\tfrac{1+\sqrt{5}}{2}\right)^{n-2}\left(\tfrac{6+2\sqrt{5}}{4}\right) - \tfrac{1}{\sqrt{5}}\left(\tfrac{1-\sqrt{5}}{2}\right)^{n-2}\left(\tfrac{6-2\sqrt{5}}{4}\right) \\
&= \tfrac{1}{\sqrt{5}}\left(\tfrac{1+\sqrt{5}}{2}\right)^{n-2}\left(\tfrac{1+\sqrt{5}}{2}\right)^2 - \tfrac{1}{\sqrt{5}}\left(\tfrac{1-\sqrt{5}}{2}\right)^{n-2}\left(\tfrac{1-\sqrt{5}}{2}\right)^2 = F_n\,.
\end{aligned}$$

□

Although our inductive proof certifies the correctness of the Binet Formula 0.3.4, it fails to provide any insight into its source. By mimicking a method for solving linear differential equations with constant coefficients, we begin to address this deficiency. We explore these ideas more fully after developing the theory of generating functions.

Alternative Proof of the Binet Formula 0.3.4. Suppose that the recurrence $F_n - F_{n-1} - F_{n-2} = 0$ has a solution of the form $F_n = x^n$ for some real number x. It would follow that

$$0 = x^n - x^{n-1} - x^{n-2} = x^{n-2}(x^2 - x - 1)$$

so the number x is either $(1-\sqrt{5})/2$, 0, or $(1+\sqrt{5})/2$. Since the recurrence is linear, the general solution has the form

$$c_1\left(\frac{1+\sqrt{5}}{2}\right)^n + c_2\left(\frac{1-\sqrt{5}}{2}\right)^n$$

for some constants c_1 and c_2. The initial conditions yield the equations $0 = c_1 + c_2$ and $1 = c_1((1+\sqrt{5})/2) + c_2((1-\sqrt{5})/2)$, so $c_1 = -c_2$ and

$$1 = c_1\left[\left(\tfrac{1+\sqrt{5}}{2}\right) - \left(\tfrac{1-\sqrt{5}}{2}\right)\right] = c_1\left(\tfrac{1+\sqrt{5}-1+\sqrt{5}}{2}\right) = \sqrt{5}\,c_1\,.$$

Therefore, the solution to the initial value problem is

$$F_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n . \qquad \square$$

Exercises

Problem 0.3.5. For all $n \in \mathbb{N}$, let F_n denote the n-th Fibonacci number.

(i) For all $n \geqslant 1$, show that $\begin{bmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n$.

(ii) Use matrix multiplication to prove that, for all $m, n \in \mathbb{N}$, we have $F_{m+n} = F_{m+1} F_n + F_m F_{n-1}$.

Problem 0.3.6. Show that $\lim\limits_{n\to\infty} \frac{F_{n+1}}{F_n} = \frac{1+\sqrt{5}}{2}$.

The Golden ratio is the irrational number $(1+\sqrt{5})/2 \approx 1.6180339\ldots$.

Problem 0.3.7. The ***Fibonacci polynomials*** are the univariate polynomials defined by $F_0(x) := 0$, $F_1(x) := 1$, and

$$F_n(x) := x\,F_{n-1}(x) + F_{n-2}(x) \quad \text{for all } n \geqslant 2.$$

If $\alpha(x) := (x + \sqrt{x^2+4})/2$ and $\beta(x) := (x - \sqrt{x^2+4})/2$, then prove that, for all $n \in \mathbb{N}$, we have

$$F_n(x) = \frac{\big(\alpha(x)\big)^n + \big(\beta(x)\big)^n}{\alpha(x) - \beta(x)} .$$

Problem 0.3.8. The ***Pell numbers*** are the integers defined by $P_0 := 0$, $P_1 := 1$, and $P_n := 2\,P_{n-1} + P_{n-2}$ for all $n \geqslant 2$. The first numbers are $0, 1, 2, 5, 12, 29, 70, 169, 408, 985, 2378, 5741, \ldots$.

(i) For all $n \geqslant 1$, show that $\begin{bmatrix} P_{n+1} & P_n \\ P_n & P_{n-1} \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}^n$.

(ii) For all $n \geqslant 1$, prove that $P_n^2 - P_{n+1} P_{n-1} = (-1)^{n+1}$.

(iii) By diagonalizing the matrix from part (i), derive a Binet-type formula for the Pell numbers.

Problem 0.3.9. For two fixed integers p and q, the ***Lucas sequence of the first kind*** is defined by $L_0 := 0$, $L_1 := 1$, and $L_n := p\,L_{n-1} + q\,L_{n-2}$ for all $n \geqslant 2$. For all $n, k \in \mathbb{N}$ satisfying $0 \leqslant k \leqslant n$, the ***Lusasnomial coefficients*** are defined as

$$\binom{n}{k}_{\mathrm{L}} := \frac{L_n\,L_{n-1}\,L_{n-2}\cdots L_{n-k+1}}{L_k\,L_{k-1}\,L_{k-2}\cdots L_1} = \prod_{j=1}^{k} \frac{L_{n-k+j}}{L_j} .$$

(i) Prove the symmetry identity $\binom{n}{k}_{\mathrm{L}} = \binom{n}{n-k}_{\mathrm{L}}$.

(ii) Prove the additive identity

$$\binom{n}{k}_{\mathrm{L}} = L_{k+1}\binom{n-1}{k}_{\mathrm{L}} + q\,L_{n-k-1}\binom{n-1}{k-1}_{\mathrm{L}} .$$

(iii) Compute the (5×5)-matrix whose (n,k)-entry is $\binom{n}{k}_{\mathrm{L}}$.