Problem Set #4

Due: Friday, 28 February 2020

1. Prove that the polytabloids corresponding to the list $\left(\begin{array}{|c|c|c|}\hline 1&3&4\\\hline 2\\\cline{1-1}\end{array}, \begin{array}{|c|c|c|}\hline 1&2&4\\\hline 3\\\cline{1-1}\end{array}, \begin{array}{|c|c|c|}\hline 1&2&3\\\hline 4\\\cline{1-1}\end{array}\right)$ form an ordered basis for $S^{(3,1)}$, and compute the matrices of the adjacent transpositions relative to this ordered basis.

2. Prove that the polytabloids corresponding to the list

$$\left(\begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1} 4\\\cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1} 4\\\cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 1&4\\\hline 2\\\cline{1-1} 3\\\cline{1-1}\end{array}\right)$$

form an ordered basis for $S^{(2,1^2)}$, and compute the matrices of the adjacent transpositions relative to this ordered basis.

3. **(a)** Prove that the polytabloids corresponding to $\left(\begin{array}{|c|c|}\hline 1&2\\\hline 3&4\\\hline\end{array}, \begin{array}{|c|c|}\hline 1&3\\\hline 2&4\\\hline\end{array}\right)$ form an ordered basis for $S^{(2,2)}$.

(b) Show that $M^{(2,2)} \cong S^{(2,2)} \oplus S^{(3,1)} \oplus S^{(4)}$.

Hint. Consider the following elements of $M^{(2,2)}$;

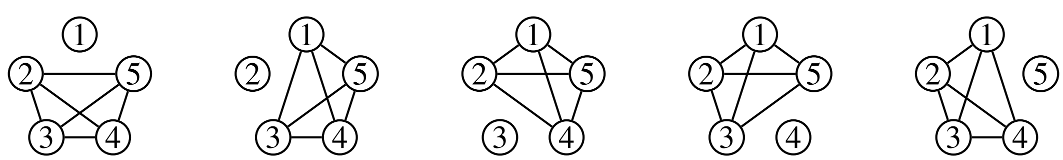
$$\vec{\mathbf{u}}_1 := \begin{array}{|cc|}\hline 1&2\\\hline 3&4\\\hline\end{array} + \begin{array}{|cc|}\hline 1&3\\\hline 2&4\\\hline\end{array} + \begin{array}{|cc|}\hline 1&4\\\hline 2&3\\\hline\end{array} + \begin{array}{|cc|}\hline 2&3\\\hline 1&4\\\hline\end{array} + \begin{array}{|cc|}\hline 2&4\\\hline 1&3\\\hline\end{array} + \begin{array}{|cc|}\hline 3&4\\\hline 1&2\\\hline\end{array},$$

$$\vec{\mathbf{v}}_1 := \begin{array}{|cc|}\hline 1&2\\\hline 3&4\\\hline\end{array} - \begin{array}{|cc|}\hline 1&4\\\hline 2&3\\\hline\end{array} - \begin{array}{|cc|}\hline 2&3\\\hline 1&4\\\hline\end{array} + \begin{array}{|cc|}\hline 3&4\\\hline 1&2\\\hline\end{array}, \qquad \vec{\mathbf{v}}_2 := \begin{array}{|cc|}\hline 1&2\\\hline 3&4\\\hline\end{array} - \begin{array}{|cc|}\hline 1&3\\\hline 2&4\\\hline\end{array} - \begin{array}{|cc|}\hline 2&4\\\hline 1&3\\\hline\end{array} + \begin{array}{|cc|}\hline 3&4\\\hline 1&2\\\hline\end{array},$$

$$\vec{\mathbf{w}}_1 := \begin{array}{|cc|}\hline 1&2\\\hline 3&4\\\hline\end{array} - \begin{array}{|cc|}\hline 3&4\\\hline 1&2\\\hline\end{array}, \qquad \vec{\mathbf{w}}_2 := \begin{array}{|cc|}\hline 1&3\\\hline 2&4\\\hline\end{array} - \begin{array}{|cc|}\hline 2&4\\\hline 1&3\\\hline\end{array}, \qquad \vec{\mathbf{w}}_3 := \begin{array}{|cc|}\hline 1&4\\\hline 2&3\\\hline\end{array} - \begin{array}{|cc|}\hline 2&3\\\hline 1&4\\\hline\end{array}.$$

We claim that the vector spaces $U := \operatorname{Span}(\vec{\mathbf{u}}_1)$, $V := \operatorname{Span}(\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2)$, and $W := \operatorname{Span}(\vec{\mathbf{w}}_1, \vec{\mathbf{w}}_2, \vec{\mathbf{w}}_3)$ are relevant $\mathbb{C}[\mathfrak{S}_4]$-submodules of $M^{(2,2)}$.

4. Show that the given graphs correspond to a basis of $(S^{(3,2)})^{\perp} \subset M^{(3,2)}$.



5. Fix $n \in \mathbb{N}$ such that $n \geqslant 2$. Let $\chi_{(n-1,1)}$ denote the character of the Specht module $S^{(n-1,1)}$. Prove that the value of $\chi_{(n-1,1)}$ on a permutation $\sigma \in \mathfrak{S}_n$ is one less than the number of fixed points (or cycles of length 1) in σ.

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