Problem Set #4

Due: Friday, February 2, 2007

1. Let $L\colon C^2([0,1]) \to C([0,1])$ be defined by $Lf = f''$.

(a) Show that L has no left inverses. *Hint*: L is not injective.

(b) Show that the operators G_1 and G_2, defined as follows, are right inverses:

$$(G_1 f)(x) = \int_0^x (x-t) f(t)\, dt\,,$$

$$(G_2 f)(x) = \int_0^1 g(x,y) f(y)\, dy\,, \qquad \text{where } g(x,y) = \begin{cases} x(y-1) & x < y \\ y(x-1) & y \le x\,. \end{cases}$$

(c) Let U_1 be the set of functions in $C^2([0,1])$ satisfying $f(0) = f'(0) = 0$. Show that $G_1 = L^{-1}$ if the domain of L is restricted to U_1.

(d) Let U_2 be the set of functions in $C^2([0,1])$ satisfying $f(0) = f(1) = 0$. Show that $G_2 = L^{-1}$ if the domain of L is restricted to U_2.

2. Let V be a finite dimensional vector space and consider $S, T \in \operatorname{End}(V)$.

(a) Show that ST is invertible if and only if both S and T are invertible.

(b) Prove that $ST = I$ if and only if $TS = I$.

(c) Give an example illustrating that both (a) and (b) are false over an infinite dimensional vector space.

3. Define $J\colon \mathbb{R}[t]_{\le 2} \to \mathbb{R}[t]_{\le 2}$ by $(Jp)(t) = \dfrac{1}{2}\displaystyle\int_{-1}^{1} (3 + 6st - 15s^2t^2) p(s)\, ds$.

(a) Find the matrix $\mathcal{M}(J)$ with respect to the basis $(1, t, t^2)$.

(b) Find a basis for $\operatorname{Ker} J$ and $\operatorname{Im} J$.

(c) Show that J^{-1} exists and find an expression for $J^{-1}(a + bt + ct^2)$.

(d) Find p such that $J(p) = (1+t)^2$.

(e) Find q such that $J^2(q) = t^2$.