Jordan Canonical Form

Given $d \in \mathbb{N}$ and $\lambda \in \mathbb{K}$, the *Jordan block* $J_d(\lambda)$ is the upper-triangular $(d \times d)$-matrix

$$J_d(\lambda) := \begin{bmatrix} \lambda & 1 & 0 & \cdots & 0 & 0 \\ 0 & \lambda & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda & 1 \\ 0 & 0 & 0 & \cdots & 0 & \lambda \end{bmatrix}$$

The scalar λ appears d times on the main diagonal and $+1$ appears $(d-1)$ times on the super-diagonal. All other entries are zero. Note $J_1(\lambda) = [\lambda]$.

Theorem. *Let $T \in \operatorname{End}(V)$ and $n = \dim V$. If the minimal polynomial of T is the product of linear factors over $\mathbb{K}$, then there exists a basis of V such that*

$$\mathcal{M}(T) = J = \begin{bmatrix} J_{d_1}(\lambda_1) & 0 & \cdots & 0 \\ 0 & J_{d_2}(\lambda_2) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & J_{d_m}(\lambda_m) \end{bmatrix}$$

where $d_1 + \cdots + d_m = n$. The matrix J is unique up to the order of the diagonal blocks.

Observations/Facts.

(a) The number m of Jordan blocks (counting multiple occurrences of the same block) is the number of linearly independent eigenvectors of J.

(b) The matrix J is diagonalizable if and only if $m = n$.

(c) The number of Jordan blocks corresponding to a given eigenvalue is the geometric multiplicity of the eigenvalue, which is the dimension of the associated eigenspace.

(d) The sum of the orders of all the Jordan blocks corresponding to a given eigenvalue is the algebraic multiplicity of the eigenvalue.

(e) A Jordan matrix is *not* completely determined in general by a knowledge of the eigenvalues and the dimension of their generalized and standard eigenspaces. One must also know the sizes of the Jordan blocks corresponding to each eigenvalue.

(f) The size of the largest Jordan block corresponding to an eigenvalue λ is the multiplicity of λ as a root of the minimal polynomial.

(g) The sizes of the Jordan blocks corresponding to a given eigenvalue are determined by a knowledge of the ranks of certain powers. For example, if

$$J = \begin{bmatrix} \begin{smallmatrix} 2&1&0\\0&2&1\\0&0&2 \end{smallmatrix} & & & \\ & \begin{smallmatrix} 2&1\\0&2 \end{smallmatrix} & & \\ & & \begin{smallmatrix} 2&1\\0&2 \end{smallmatrix} & \\ & & & 2 \end{bmatrix} \quad \text{then} \quad J - 2I = \begin{bmatrix} \begin{smallmatrix} 0&1&0\\0&0&1\\0&0&0 \end{smallmatrix} & & & \\ & \begin{smallmatrix} 0&1\\0&0 \end{smallmatrix} & & \\ & & \begin{smallmatrix} 0&1\\0&0 \end{smallmatrix} & \\ & & & 0 \end{bmatrix} \quad (J-2I)^2 = \begin{bmatrix} \begin{smallmatrix} 0&0&1\\0&0&0\\0&0&0 \end{smallmatrix} & & & \\ & \begin{smallmatrix} 0&0\\0&0 \end{smallmatrix} & & \\ & & \begin{smallmatrix} 0&0\\0&0 \end{smallmatrix} & \\ & & & 0 \end{bmatrix}$$

and $(J-2I)^3 = 0$. Thus we know that

$$(J-2I)^3 = 0 \qquad \operatorname{rank}(J-2I)^2 = 1 \qquad \operatorname{rank}(J-2I) = 4\,.$$

This list of numbers is sufficient to determine the block structure of J. The fact that $(J-2I)^3$ tells us that the largest block has order 3. The rank of $(J-2I)^2$ will be the number of blocks of order 3, so there is only one. The rank of $(J-2I)$ is the twice the number of blocks of order 3 plus the number of blocks of order 2, so there are two of them. The number of blocks of order 1 is $8 - (2 \times 2) - 3 = 1$. A similar procedure can be applied to direct sums of Jordan blocks of any size.