Problem Set #4

Due: 8 October 2010

1. **(a)** Let $\vec{\boldsymbol{e}}_1, \dots, \vec{\boldsymbol{e}}_n$ be the standard basis of $\mathbb{R}^n$. If $\vec{\boldsymbol{F}}, \vec{\boldsymbol{G}} \colon \mathbb{R}^n \to \mathbb{R}^3$ are differentiable at $\vec{\boldsymbol{a}} \in \mathbb{R}^n$, then show that, for $1 \le i \le n$, we have

$$\big[D(\vec{\boldsymbol{F}} \times \vec{\boldsymbol{G}})(\vec{\boldsymbol{a}})\big]\vec{\boldsymbol{e}}_i = \big[D\vec{\boldsymbol{F}}(\vec{\boldsymbol{a}})\big]\vec{\boldsymbol{e}}_i \times \vec{\boldsymbol{G}}(\vec{\boldsymbol{a}}) + \vec{\boldsymbol{F}}(\vec{\boldsymbol{a}}) \times \big[D\vec{\boldsymbol{G}}(\vec{\boldsymbol{a}})\big]\vec{\boldsymbol{e}}_i \,.$$

(b) Suppose that $n = 3$, $\vec{\boldsymbol{F}}(\vec{\boldsymbol{a}}) = 2\vec{\boldsymbol{\imath}} + \vec{\boldsymbol{\jmath}} + 2\vec{\boldsymbol{k}}$, $\vec{\boldsymbol{G}}(\vec{\boldsymbol{a}}) = \vec{\boldsymbol{\imath}} + 2\vec{\boldsymbol{\jmath}} + \vec{\boldsymbol{k}}$,

$$D\vec{\boldsymbol{F}}(\vec{\boldsymbol{a}}) = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \quad \text{and} \quad D\vec{\boldsymbol{G}}(\vec{\boldsymbol{a}}) = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 1 \\ 1 & 0 & -1 \end{bmatrix}.$$

Find $\big[D(\vec{\boldsymbol{F}} \times \vec{\boldsymbol{G}})(\vec{\boldsymbol{a}})\big](\vec{\boldsymbol{\imath}} + \vec{\boldsymbol{\jmath}} + \vec{\boldsymbol{k}})$.

2. **(a)** Two surfaces are said to be ***orthogonal*** to each other at a point P if the normals to their tangent planes are perpendicular at P. Show that the surfaces

$$z = \tfrac{1}{2}(x^2 + y^2 - 1) \quad \text{and} \quad z = \tfrac{1}{2}(1 - x^2 - y^2)$$

are orthogonal at all points of intersection.

(b) Show that the Laplacian operator $\nabla^2 := \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2} + \dfrac{\partial^2}{\partial z^2}$ in $\mathbb{R}^3$ is given in cylindrical coordinates by the formula

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}\,.$$

3. Consider the function

$$f(x,y) = \begin{cases} \dfrac{xy^2}{x^2 + y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

(a) Find the partial derivatives $f_x(0,0)$ and $f_y(0,0)$.

(b) If $\vec{\boldsymbol{H}} \colon \mathbb{R} \to \mathbb{R}^2$ is defined by $\vec{\boldsymbol{H}}(t) = at\vec{\boldsymbol{\imath}} + bt\vec{\boldsymbol{\jmath}}$ for constants a and b, then show that $f \circ \vec{\boldsymbol{H}}$ is differentiable and find $D(f \circ \vec{\boldsymbol{H}})(0)$.

(c) Calculate $Df(0,0)D\vec{\boldsymbol{H}}(0)$. How can this answer be reconciled with the answer in part (b) and the chain rule?