Problem Set #12

Due: 28 November 2008

1. **(a)** Show that the path $\vec{\boldsymbol{\gamma}}\colon [0, 2\pi] \to \mathbb{R}^3$ defined by $\vec{\boldsymbol{\gamma}}(t) := \cos(t)\vec{\boldsymbol{\imath}} + \sin(t)\vec{\boldsymbol{\jmath}} + \sin(2t)\vec{\boldsymbol{k}}$ lies on the surface $z = 2xy$.

(b) Evaluate $\int_C \big(y^3 + \cos(x)\big)dx + \big(\sin(y) + z^2\big)dy + x\,dz$ where C is the closed curve parametrized by $\vec{\boldsymbol{\gamma}}$.

2. **(a)** Evaluate the circulation of the vector field $\vec{\boldsymbol{G}}(x, y, z) := xy\vec{\boldsymbol{\imath}} + z\vec{\boldsymbol{\jmath}} + 3y\vec{\boldsymbol{k}}$ around a square of side length 6, centered at the origin lying in the yz-plane, and oriented counterclockwise viewed from the positive x-axis.

(b) Let $\vec{\boldsymbol{H}}(x, y, z) := (y - z)\vec{\boldsymbol{\imath}} + (x + z)\vec{\boldsymbol{\jmath}} + xy\vec{\boldsymbol{k}}$ and let C be the circle of radius 3 centered at $(2, 1, 0)$ in the xy-plane oriented counterclockwise when viewed from above. Compute $\int_C \vec{\boldsymbol{H}} \cdot d\vec{\boldsymbol{r}}$. Is $\vec{\boldsymbol{H}}$ path-independent? Explain.

3. Water in a bathtub has velocity vector field[1] near the drain given, for x, y, z in cm, by

$$\vec{\boldsymbol{V}}(x, y, z) := \frac{-y\vec{\boldsymbol{\imath}} + x\vec{\boldsymbol{\jmath}}}{(z^2+1)^2} + \frac{-z(x\vec{\boldsymbol{\imath}} + y\vec{\boldsymbol{\jmath}})}{(z^2+1)^2} - \frac{\vec{\boldsymbol{k}}}{z^2+1} = -\frac{y + xz}{(z^2+1)^2}\vec{\boldsymbol{\imath}} - \frac{yz - x}{(z^2+1)^2}\vec{\boldsymbol{\jmath}} - \frac{1}{z^2+1}\vec{\boldsymbol{k}}\ \text{cm}\cdot\text{s}^{-1}\,.$$

(a) The drain in the bathtub is a disk in the xy-plane with center at the origin and radius 1 cm. Find the rate at which the water is leaving the bathtub.

(b) Find the divergence of $\vec{\boldsymbol{V}}$.

(c) Find the flux of the water through the hemisphere of radius 1, centered at the origin, lying below the xy-plane and oriented downward.

(d) Consider the vector field

$$\vec{\boldsymbol{U}}(x, y, z) := \frac{1}{2}\left(\frac{y}{z^2+1}\vec{\boldsymbol{\imath}} - \frac{x}{z^2+1}\vec{\boldsymbol{\jmath}} - \frac{x^2+y^2}{(z^2+1)^2}\vec{\boldsymbol{k}}\right).$$

Compute $\int_E \vec{\boldsymbol{U}} \cdot d\vec{\boldsymbol{r}}$ where E is the edge of the drain oriented clockwise when viewed from above.

(e) Calculate $\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{U}}$.

(f) Explain why your answers in parts (a) and (d) are equal.

---

[1] The denominators $(z^2+1)^2$ and z^2+1 are always positive and so affect the magnitude (but not the direction) of the motion. The $(-y\vec{\boldsymbol{\imath}} + x\vec{\boldsymbol{\jmath}})$ term represents rotation around the z-axis (counterclockwise when viewed from above). The $-z(x\vec{\boldsymbol{\imath}} + y\vec{\boldsymbol{\jmath}})$ term represents radial motion (towards the z-axis when $z > 0$ and away when $z < 0$). The $\vec{\boldsymbol{k}}$ term is downward motion. Hence $\vec{\boldsymbol{V}}$ is a flow rotating inward and downward around the z-axis (for $z > 0$) like an actual bathtube drain.