# Problems 07

Due: Friday, 3 March 2023 before 17:00 EST

**P7.1.** Let $m$ and $n$ be positive integers. When $m$ divides $n$, confirm that there exists a ring homomorphism from $\mathbb{Z}/\langle n\rangle$ to $\mathbb{Z}/\langle m\rangle$.

**P7.2.** Let $\mathrm{U}_3(\mathbb{Z})$ be the subset of all upper triangular $(3 \times 3)$-matrices with integer entries;

$$\mathrm{U}_3(\mathbb{Z}) := \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ 0 & a_4 & a_5 \\ 0 & 0 & a_6 \end{bmatrix} \,\middle|\, a_1, a_2, \dots, a_6 \in \mathbb{Z} \right\}.$$

**(i)** Verify that $\mathrm{U}_3(\mathbb{Z})$ is a subring of the ring of all $(3\times3)$-matrices with integer entries.

**(ii)** Given the matrix

$$\mathbf{N} := \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$

let $\eta\colon \mathbb{Z}[x] \to \mathrm{U}_3(\mathbb{Z})$ be the ring homomorphism defined by

$$\eta(a_m x^m + a_{m-1} x^{m-1} + \cdots + a_1 x + a_0) = a_m \mathbf{N}^m + a_{m-1} \mathbf{N}^{m-1} + \cdots + a_1 \mathbf{N} + a_0 \mathbf{I}.$$

Find a polynomial $g$ in $\mathbb{Z}[x]$ such that $\mathrm{Ker}(\eta) = \langle g \rangle$.

**P7.3.** Consider the ideal $I := \langle 1 + 2\,\mathrm{i} \rangle$ in the ring $\mathbb{Z}[\mathrm{i}] := \{a + b\,\mathrm{i} \in \mathbb{C} \mid a, b \in \mathbb{Z}\}$ of Gaussian integers. Let $R := \mathbb{Z}[\mathrm{i}]/I$ be the quotient ring.

**(i)** Are the cosets $\mathrm{i} + I$ and $2 + I$ equal in $R$?

**(ii)** Are the cosets $4 + I$ and $-1 + I$ equal in $R$?

**(iii)** How many elements does $R$ have?

**(iv)** Is $R$ a field?