Problems 23

Due: Friday, 1 April 2022 before 17:00 EST

P23.1. Consider

$$\mathbf{A} := \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & -1 & 1 & -1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} \qquad \text{and} \qquad \boldsymbol{b} := \begin{bmatrix} 5 \\ 3 \\ -1 \\ 1 \end{bmatrix}.$$

Show that a least-squares solution to $\mathbf{A}\vec{\mathbf{x}} = \vec{\mathbf{b}}$ is not unique and solve the normal equations to find all of the least-squares solutions.

P23.2. The population of Canada, as determined by the Canadian census, was as follows:

year	2001	2006	2011	2016	2021
population (in millions)	30.0	31.6	33.5	35.2	37.0

Let t denote the time measured in years from 2001.

(i) Suppose that population of Canada (measure in millions) is modeled by the linear function $p_\ell(t) = mt + b$. Find the least-squares estimates for the parameters m and b.

(ii) Suppose that the population of Canada (measure in millions) is modeled by the exponential function $p_e(t) = c\,e^{\lambda t}$. Linearize the model and use the least-squares method to estimate the parameters c and λ.

P23.3. Fix a nonnegative integer n. Consider the finite set

$$\mathcal{X} := \left\{0, \tfrac{2\pi}{n}, \tfrac{4\pi}{n}, \dots, \tfrac{2\pi(n-1)}{n}\right\} = \left\{\tfrac{2\pi\ell}{n} \in \mathbb{R} \;\middle|\; 0 \leqslant \ell \leqslant n-1\right\}$$

of n real numbers corresponding the left endpoints of equally spaced intervals between 0 and 2π. Let $V := \mathbb{C}^{\mathcal{X}}$ the complex inner product space, consisting of all functions from the finite set $\mathcal{X}$ to $\mathbb{C}$ equipped with the inner product

$$\langle f, g\rangle := \sum_{x \in \mathcal{X}} f(x)\,\overline{g(x)} = \sum_{\ell=0}^{n-1} f\big(\tfrac{2\pi\ell}{n}\big)\,\overline{g\big(\tfrac{2\pi\ell}{n}\big)}.$$

(i) For all integers j satisfying $0 \leqslant j \leqslant n-1$, show that the functions $w_j(x) := \exp(-j\,x\,\mathrm{i})$ are pairwise orthogonal and compute $\|w_j(x)\|$.

(ii) For all integers k satisfying $0 \leqslant k \leqslant n-1$, consider the indicator function

$$h_k(x) := \begin{cases} 1 & \text{if } x = \frac{2\pi k}{n} \\ 0 & \text{if } x \neq \frac{2\pi k}{n} \end{cases}.$$

Which function in the linear subspace $W := \operatorname{Span}\big(w_0(x), w_1(x), \dots, w_{n-1}(x)\big) \subset V$ is the best approximation the function $h_k(x)$?

(iii) For all integers k satisfying $0 \leqslant k \leqslant n-1$, calculate the norm of the different between $h_k(x)$ and its best approximate.