Problems 22

Due: Friday, 25 March 2022 before 17:00 EST

P22.1. Fix $n \coloneqq 2$. Consider the $\mathbb{R}$-vector space $V \coloneqq \mathbb{R}[t]_{\leqslant n}$ with the inner product defined, for all polynomials f and g in $\mathbb{R}[t]_{\leqslant n}$, by

$$\langle f, g \rangle \coloneqq \int_0^\infty f(x)\, g(x)\, e^{-x}\, dx\,.$$

(i) Apply the orthonormalization algorithm to the monomial basis $(1, t, t^2, \dots, t^n)$ to produce the orthonormal basis $(L_0, L_1, \dots, L_n)$ of the inner product space V.

(ii) For any $0 \leqslant k \leqslant n$, consider the linear operator $D_k \colon V \to V$ defined, for all polynomials f in V, by $D_k[f] \coloneqq t\, f''(t) + (1-t)\, f'(t) + k\, f(t)$. Show that $\operatorname{Span}(L_k) = \operatorname{Ker}(D_k)$.

P22.2. Using the properties of a projection operator, determine all of its eigenvalues and describe the corresponding eigenspaces.

P22.3. Equipe the coordinate space $\mathbb{R}^5$ with the weighted (or non-standard) inner product defined, for all vectors $\boldsymbol{u} \coloneqq \begin{bmatrix} u_1 & u_2 & u_3 & u_4 & u_5 \end{bmatrix}^\mathsf{T}$ and $\boldsymbol{v} \coloneqq \begin{bmatrix} v_1 & v_2 & v_3 & v_4 & v_5 \end{bmatrix}^\mathsf{T}$, by

$$\langle \boldsymbol{u}, \boldsymbol{v} \rangle \coloneqq \frac{1}{60}(u_1 v_1 + 15 u_2 v_2 + 20 u_3 v_3 + 12 u_4 v_4 + 12 u_5 v_5)\,.$$

(i) Verify that the three vectors $\boldsymbol{w}_1 \coloneqq \begin{bmatrix} 8 & -4 & -1 & -7 & -7 \end{bmatrix}^\mathsf{T}$, $\boldsymbol{w}_2 \coloneqq \begin{bmatrix} 6 & -2 & 0 & 1+\sqrt{5} & 1-\sqrt{5} \end{bmatrix}^\mathsf{T}$, and $\boldsymbol{w}_3 \coloneqq \begin{bmatrix} 5 & 1 & -1 & 0 & 0 \end{bmatrix}^\mathsf{T}$ are pairwise orthogonal.

(ii) Calculate the norms of the vectors $\boldsymbol{w}_1$, $\boldsymbol{w}_2$, and $\boldsymbol{w}_3$.

(iii) Compute the orthogonal projection of the vector $\boldsymbol{z} \coloneqq \begin{bmatrix} 11 & 3 & -4 & -9 & -9 \end{bmatrix}^\mathsf{T}$ onto the linear subspace $W \coloneqq \operatorname{Span}(\boldsymbol{w}_1, \boldsymbol{w}_2, \boldsymbol{w}_3)$.