Problems 18

Due: Friday, 18 February 2022 before 17:00 EST

P18.1. Consider the three complex (2×2)-matrices

$$\mathbf{X} := \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad \mathbf{H} := \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \qquad \text{and} \qquad \mathbf{Y} := \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

Problem C16.2 shows that $\mathcal{B} := (\mathbf{X}, \mathbf{H}, \mathbf{Y})$ is a basis for the linear subspace $\mathfrak{sl}(2, \mathbb{C})$ of traceless complex (2×2)-matrices. For a fixed complex (2×2)-matrix $\mathbf{A}$, let $\mathrm{ad}_{\mathbf{A}} \colon \mathfrak{sl}(2, \mathbb{C}) \to \mathbb{C}^{2 \times 2}$ be the map defined, for all matrices $\mathbf{B}$ in $\mathfrak{sl}(2, \mathbb{C})$, by $\mathrm{ad}_{\mathbf{A}}(\mathbf{B}) := \mathbf{A}\,\mathbf{B} - \mathbf{B}\,\mathbf{A}$.

(i) Show that $\mathrm{ad}_{\mathbf{A}}$ is linear.
(ii) Show that the image of $\mathrm{ad}_{\mathbf{A}}$ is contained in $\mathfrak{sl}(2, \mathbb{C})$.
(iii) Determine the matrices $(\mathrm{ad}_{\mathbf{X}})_{\mathcal{B}}^{\mathcal{B}}$, $(\mathrm{ad}_{\mathbf{H}})_{\mathcal{B}}^{\mathcal{B}}$, and $(\mathrm{ad}_{\mathbf{Y}})_{\mathcal{B}}^{\mathcal{B}}$.

P18.2. Let $J \colon \mathbb{R}[t]_{\leqslant 2} \to \mathbb{R}[t]_{\leqslant 2}$ be the linear operator defined, for all polynomials f in $\mathbb{R}[t]_{\leqslant 2}$ by

$$J[f] := \tfrac{1}{2} \int_{-1}^{1} (3 + 6\,st - 15\,s^2 t^2)\, f(s)\, ds\,.$$

(i) Let $\mathcal{M} := (1, t, t^2)$ denote the monomial basis of $\mathbb{R}[t]_{\leqslant 2}$. Compute the matrix $(J)_{\mathcal{M}}^{\mathcal{M}}$.
(ii) Find bases for $\mathrm{Ker}(J)$ and $\mathrm{Im}(J)$.
(iii) Show that J^{-1} exists and find an expression for $J^{-1}[a + b\,t + c\,t^2]$.
(iv) Find f such that $J[f] = (1 + t)^2$.
(v) Find g such that $J^2[g] = t^2$.

P18.3. Consider two square matrices $\mathbf{A}$ and $\mathbf{B}$ that are similar.

(i) Prove by induction that, for all nonnegative integers m, the matrices $\mathbf{A}^m$ and $\mathbf{B}^m$ are similar.
(ii) For any polynomial f, show that $f(\mathbf{A})$ and $f(\mathbf{B})$ are similar.