Problems 09

Due: Friday, 12 November 2021 before 17:00 EDT

1. Consider the matrix

$$\mathbf{P} := \begin{bmatrix} -1 & -1 & -3 \\ 2 & -3 & 2 \\ 1 & -1 & 2 \end{bmatrix}.$$

(i) Demonstrate, via direct computation, that $\mathbf{P}^3 + 2\,\mathbf{P}^2 + 2\,\mathbf{P} - 3\,\mathbf{I} = \mathbf{0}$.

(ii) Calculate $\mathbf{Q} := \frac{1}{3}(\mathbf{P}^2 + 2\,\mathbf{P} + 2\,\mathbf{I})$ and verify that $\mathbf{Q} = \mathbf{P}^{-1}$.

(iii) Explain how $\mathbf{Q}$ in part (ii) can be obtained from the equation in part (i).

2. Fix two positive integers m and n.

(i) Let $\mathbf{M}$ be an invertible $(m \times m)$-matrix, let $\mathbf{N}$ be an invertible $(n \times n)$-matrix, let $\mathbf{P}$ be an $(m \times n)$-matrix, and let $\mathbf{Q}$ be an $(n \times m)$-matrix. Verify the ***Woodbury matrix identity***:

$$(\mathbf{M} + \mathbf{P}\,\mathbf{N}\,\mathbf{Q})^{-1} = \mathbf{M}^{-1} - \mathbf{M}^{-1}\,\mathbf{P}(\mathbf{N}^{-1} + \mathbf{Q}\,\mathbf{M}^{-1}\,\mathbf{P})^{-1}\,\mathbf{Q}\,\mathbf{M}^{-1}.$$

(ii) Let $\mathbf{A}$ be an invertible $(m \times m)$-matrix, let $\mathbf{B}$ be an $(n \times m)$-matrix, let $\mathbf{C}$ be an $(m \times n)$-matrix, and let $\mathbf{D}$ be an $(n \times n)$-matrix. Assuming that the ***Schur complement*** $\mathbf{S} := \mathbf{D} - \mathbf{B}\mathbf{A}^{-1}\mathbf{C}$ is invertible, establish the blockwise inversion formula:

$$\begin{bmatrix} \mathbf{A} & \mathbf{C} \\ \mathbf{B} & \mathbf{D} \end{bmatrix}^{-1} = \begin{bmatrix} \mathbf{A}^{-1} + \mathbf{A}^{-1}\,\mathbf{C}\,\mathbf{S}^{-1}\mathbf{B}\,\mathbf{A}^{-1} & -\mathbf{A}^{-1}\,\mathbf{C}\,\mathbf{S}^{-1} \\ -\mathbf{S}^{-1}\,\mathbf{B}\,\mathbf{A}^{-1} & \mathbf{S}^{-1} \end{bmatrix}.$$

3. Express $\mathbf{U} := \begin{bmatrix} -1 & 1 & -3 \\ -3 & -2 & -1 \\ -3 & 0 & 3 \end{bmatrix}$ as a product of elementary matrices.