

Taylor Series

4.1 Taylor Series and Calculation of Functions

Limit processes are the basis of calculus. For example, the derivative

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

is the limit of the difference quotient where both the numerator and the denominator go to zero. A Taylor series illustrates another type of limit process. In this case an

Table 4.1 Taylor Series Expansions for Some Common Functions

$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$	for all x
$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$	for all x
$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$	for all x
$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$	$-1 \leq x \leq 1$
$\arctan(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots$	$-1 \leq x \leq 1$
$(1+x)^p = 1 + px + \frac{p(p-1)}{2!}x^2 + \frac{p(p-1)(p-2)}{3!}x^3 + \dots$	for $ x < 1$

infinite number of terms is added together by taking the limit of certain partial sums. An important application is their use to represent the elementary functions: $\sin(x)$, $\cos(x)$, e^x , $\ln(x)$, etc. Table 4.1 gives several of the common Taylor series expansions. The partial sums can be accumulated until an approximation to the function is obtained that has the accuracy specified. Series solutions are used in the areas of engineering and physics.

We want to learn how a finite sum can be used to obtain a good approximation to an infinite sum. For illustration we shall use the exponential series in Table 4.1 to compute the number $e = e^1$, which is the base of the natural logarithm and exponential functions. Here we choose $x = 1$ and use the series

$$e^1 = 1 + \frac{1}{1!} + \frac{1^2}{2!} + \frac{1^3}{3!} + \frac{1^4}{4!} + \dots + \frac{1^k}{k!} + \dots$$

The definition for the sum of an infinite series in Section 1.1 requires that the partial sums S_N tend to a limit. The values of these sums are given in Table 4.2.

A natural way to think about the power series representation of a function is to view the expansion as the limiting case of polynomials of increasing degree. If enough terms are added, then an accurate approximation will be obtained. This needs to be made precise. What degree should be chosen for the polynomial, and how do we calculate the coefficients for the powers of x in the polynomial? Theorem 4.1 answers these questions.

Table 4.2 Partial Sums S_n Used to Determine e

n	$S_n = 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!}$
0	1.0
1	2.0
2	2.5
3	2.666666666666...
4	2.708333333333...
5	2.716666666666...
6	2.718055555555...
7	2.718253968254...
8	2.718278769841...
9	2.718281525573...
10	2.718281801146...
11	2.718281826199...
12	2.718281828286...
13	2.718281828447...
14	2.718281828458...
15	2.718281828459...

Theorem 4.1 (Taylor Polynomial Approximation). Assume that $f \in C^{N+1}[a, b]$ and $x_0 \in [a, b]$ is a fixed value. If $x \in [a, b]$, then

$$(1) \quad f(x) = P_N(x) + E_N(x),$$

where $P_N(x)$ is a polynomial that can be used to approximate $f(x)$:

$$(2) \quad f(x) \approx P_N(x) = \sum_{k=0}^N \frac{f^{(k)}(x_0)}{k!} (x - x_0)^k.$$

The error term $E_N(x)$ has the form

$$(3) \quad E_N(x) = \frac{f^{(N+1)}(c)}{(N + 1)!} (x - x_0)^{N+1}$$

for some value $c = c(x)$ that lies between x and x_0 .

Proof. The proof is left as an exercise. •

Relation (2) indicates how the coefficients of the Taylor polynomial are calculated. Although the error term (3) involves a similar expression, notice that $f^{(N+1)}(c)$ is to be evaluated at an undetermined number c that depends on the value of x . For this reason we do not try to evaluate $E_N(x)$: it is used to determine a bound for the accuracy of the approximation.

Example 4.1. Show why 15 terms are all that are needed to obtain the 13-digit approximation $e = 2.718281828459$ in Table 4.2.

Expand $f(x) = e^x$ in a Taylor polynomial of degree 15 using the fixed value $x_0 = 0$ and involving the powers $(x - 0)^k = x^k$. The derivatives required are $f'(x) = f''(x) = \dots = f^{(16)} = e^x$. The first 15 derivatives are used to calculate the coefficients $a_k = e^0/k!$ and are used to write

$$(4) \quad P_{15}(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^{15}}{15!}.$$

Setting $x = 1$ in (4) gives the partial sum $S_{15} = P_{15}(1)$. The remainder term is needed to show the accuracy of the approximation:

$$(5) \quad E_{15}(x) = \frac{f^{(16)}(c)x^{16}}{16!}.$$

Since we chose $x_0 = 0$ and $x = 1$, the value c lies between them (i.e., $0 < c < 1$), which implies that $e^c < e^1$. Notice that the partial sums in Table 4.2 are bounded above by 3. Combining these two inequalities yields $e^c < 3$, which is used in the following calculation

$$|E_{15}(1)| = \frac{|f^{(16)}(c)|}{16!} \leq \frac{e^c}{16!} < \frac{3}{16!} < 1.433844 \times 10^{-13}.$$

Therefore, all the digits in the approximation $e \approx 2.718281828459$ are correct, because the actual error (whatever it is) must be less than 2 in the thirteenth decimal place. ■

Instead of giving a rigorous proof of Theorem 4.1, we shall discuss some of the features of the approximation; the reader can look in any standard reference text on calculus for more details. For illustration, we again use the function $f(x) = e^x$ and the value $x_0 = 0$. From elementary calculus we know that the slope of the curve $y = e^x$ at the point (x, e^x) is $f'(x) = e^x$. Hence the slope at the point $(0, 1)$ is $f'(0) = 1$. Therefore, the tangent line to the curve at the point $(0, 1)$ is $y = 1 + x$. This is the same formula that would be obtained if we used $N = 1$ in Theorem 4.1; that is, $P_1(x) = f(0) + f'(0)x/1! = 1 + x$. Therefore, $P_1(x)$ is the equation of the tangent line to the curve. The graphs are shown in Figure 4.3.

Observe that the approximation $e^x \approx 1 + x$ is good near the center $x_0 = 0$ and that the distance between the curves grows as x moves away from 0. Notice that the slopes of the curves agree at $(0, 1)$. In calculus we learned that the second derivative indicates whether a curve is concave up or down. The study of curvature¹ shows that if two curves $y = f(x)$ and $y = g(x)$ have the property that $f(x_0) = g(x_0)$, $f'(x_0) = g'(x_0)$, and $f''(x_0) = g''(x_0)$ then they have the same curvature at x_0 . This property would be desirable for a polynomial function that approximates $f(x)$. Corollary 4.1 shows that the Taylor polynomial has this property for $N \geq 2$.

¹The curvature K of a graph $y = f(x)$ at (x_0, y_0) is defined by $K = |f''(x_0)|/(1+[f'(x_0)]^2)^{3/2}$.

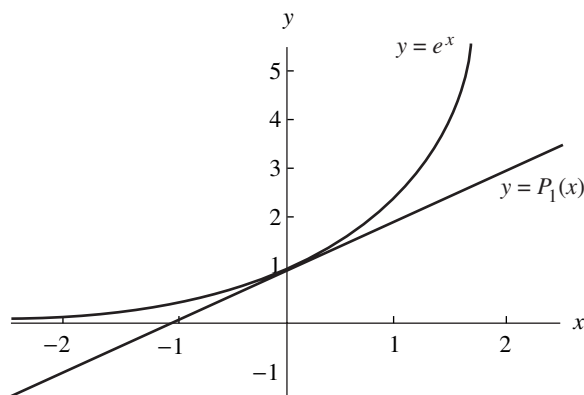


Figure 4.3 The graphs of $y = e^x$ and $y = P_1(x) = 1 + x$.

Corollary 4.1. If $P_N(x)$ is the Taylor polynomial of degree N given in Theorem 4.1, then

$$(6) \quad P_N^{(k)}(x_0) = f^{(k)}(x_0) \quad \text{for } k = 0, 1, \dots, N.$$

Proof. Set $x = x_0$ in equations (2) and (3), and the result is $P_N(x_0) = f(x_0)$. Thus statement (6) is true for $k = 0$. Now differentiate the right-hand side of (2) and get

$$(7) \quad P_N'(x) = \sum_{k=1}^N \frac{f^{(k)}(x_0)}{(k-1)!} (x-x_0)^{k-1} = \sum_{k=0}^{N-1} \frac{f^{(k+1)}(x_0)}{k!} (x-x_0)^k.$$

Set $x = x_0$ in (7) to obtain $P_N'(x_0) = f'(x_0)$. Thus statement (6) is true for $k = 1$. Successive differentiations of (7) will establish the other identities in (6). The details are left as an exercise. •

Applying Corollary 4.1, we see that $y = P_2(x)$ has the properties $f(x_0) = P_2(x_0)$, $f'(x_0) = P_2'(x_0)$, and $f''(x_0) = P_2''(x_0)$; hence the graphs have the same curvature at x_0 . For example, consider $f(x) = e^x$ and $P_2(x) = 1 + x + x^2/2$. The graphs are shown in Figure 4.4 and it is seen that they curve up in the same fashion at $(0, 1)$.

In the theory of approximation, one seeks to find an accurate polynomial approximation to the analytic function² $f(x)$ over $[a, b]$. This is one technique used in developing computer software. The accuracy of a Taylor polynomial is increased when we choose N large. The accuracy of any given polynomial will generally decrease as the value of x moves away from the center x_0 . Hence we must choose N large enough and restrict the maximum value of $|x - x_0|$ so that the error does not exceed a specified bound. If we choose the interval width to be $2R$ and x_0 in the center (i.e., $|x - x_0| < R$),

²The function $f(x)$ is analytic at x_0 if it has continuous derivatives of all orders and can be represented as a Taylor series in an interval about x_0 .

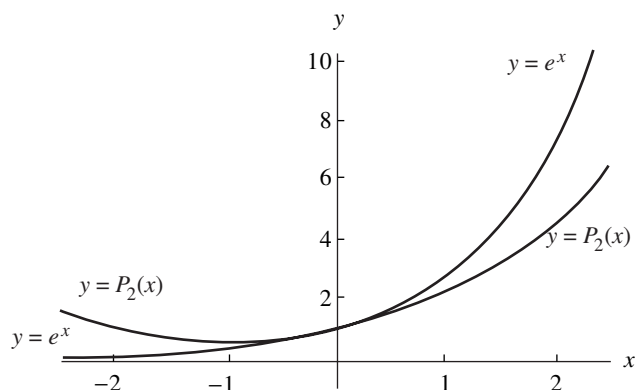


Figure 4.4 The graphs of $y = e^x$ and $y = P_2(x) = 1 + x + x^2/2$.

Table 4.3 Values for the Error Bound $|\text{error}| < e^R R^{N+1}/(N+1)!$ Using the Approximation $e^x \approx P_N(x)$ for $|x| \leq R$

	$R = 2.0,$ $ x \leq 2.0$	$R = 1.5,$ $ x \leq 1.5$	$R = 1.0,$ $ x \leq 1.0$	$R = 0.5,$ $ x \leq 0.5$
$e^x \approx P_5(x)$	0.65680499	0.07090172	0.00377539	0.00003578
$e^x \approx P_6(x)$	0.18765857	0.01519323	0.00053934	0.00000256
$e^x \approx P_7(x)$	0.04691464	0.00284873	0.00006742	0.00000016
$e^x \approx P_8(x)$	0.01042548	0.00047479	0.00000749	0.00000001

the absolute value of the error satisfies the relation

$$(8) \quad |\text{error}| = |E_N(x)| \leq \frac{MR^{N+1}}{(N+1)!},$$

where $M \leq \max\{|f^{(N+1)}(z)| : x_0 - R \leq z \leq x_0 + R\}$. If the derivatives are uniformly bounded, the error bound in (8) is proportional to $R^{N+1}/(N+1)!$ and decreases for fixed R , when N gets large or, for fixed N , when R goes to 0. Table 4.3 shows how the choices of these two parameters affect the accuracy of the approximation $e^x \approx P_N(x)$ over the interval $|x| \leq R$. The error is smallest when N is largest and R smallest. Graphs for P_2 , P_3 , and P_4 are given in Figure 4.5.

Example 4.2. Establish the error bounds for the approximation $e^x \approx P_8(x)$ on each of the intervals $|x| \leq 1.0$ and $|x| \leq 0.5$.

If $|x| \leq 1.0$, then letting $R = 1.0$ and $|f^{(9)}(c)| = |e^c| \leq e^{1.0} = M$ in (8) implies that

$$|\text{error}| = |E_8(x)| \leq \frac{e^{1.0}(1.0)^9}{9!} \approx 0.00000749.$$

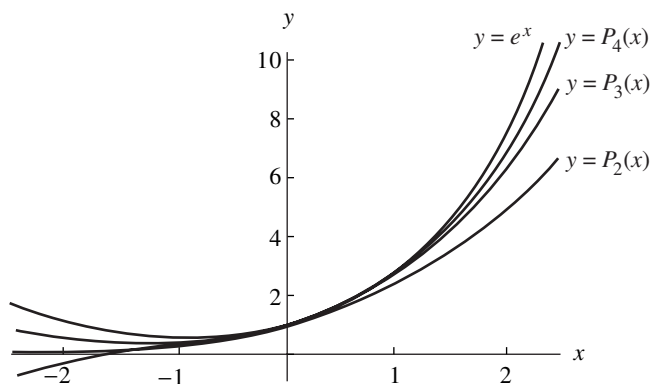


Figure 4.5 The graphs of $y = e^x$, $y = P_2(x)$, $y = P_3(x)$, and $y = P_4(x)$.

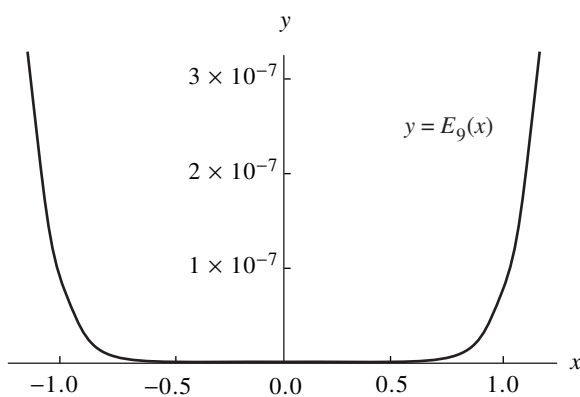


Figure 4.6 The graph of the error $y = E_9(x) = e^x - P_9(x)$.

If $|x| \leq 0.5$, then letting $R = 0.5$ and $|f^{(9)}(c)| = |e^c| \leq e^{0.5} = M$ in (8) implies that

$$|\text{error}| = |E_8(x)| \leq \frac{e^{0.5}(0.5)^9}{9!} \approx 0.00000001. \quad \blacksquare$$

Example 4.3. If $f(x) = e^x$, show that $N = 9$ is the smallest integer, so that the $|\text{error}| = |E_N(x)| \leq 0.0000005$ for x in $[-1, 1]$. Hence $P_9(x)$ can be used to compute approximate values of e^x that will be accurate in the sixth decimal place.

We need to find the smallest integer N so that

$$|\text{error}| = |E_N(x)| \leq \frac{e^c(1)^{N+1}}{(N+1)!} < 0.0000005.$$

In Example 4.2 we saw that $N = 8$ was too small, so we try $N = 9$ and discover that $|E_N(x)| \leq e^1(1)^{9+1}/(9+1)! \leq 0.000000749$. This value is slightly larger than

desired; hence we would be likely to choose $N = 10$. But we used $e^c \leq e^1$ as a crude estimate in finding the error bound. Hence 0.000000749 is a little larger than the actual error. Figure 4.6 shows a graph of $E_9(x) = e^x - P_9(x)$. Notice that the maximum vertical range is about 3×10^{-7} and occurs at the right endpoint $(1, E_9(1))$. Indeed, the maximum error on the interval is $E_9(1) = 2.718281828 - 2.718281526 \approx 3.024 \times 10^{-7}$. Therefore, $N = 9$ is justified. ■

Methods for Evaluating a Polynomial

There are several mathematically equivalent ways to evaluate a polynomial. Consider, for example, the function

$$(9) \quad f(x) = (x - 1)^8.$$

The evaluation of f will require the use of an exponential function. Or the binomial formula can be used to expand $f(x)$ in powers of x :

$$(10) \quad \begin{aligned} f(x) &= \sum_{k=0}^8 \binom{8}{k} x^{8-k} (-1)^k \\ &= x^8 - 8x^7 + 28x^6 - 56x^5 + 70x^4 - 56x^3 + 28x^2 - 8x + 1. \end{aligned}$$

Horner's method (see Section 1.1), which is also called *nested multiplication*, can now be used to evaluate the polynomial in (10). When applied to formula (10), nested multiplication permits us to write

$$(11) \quad f(x) = ((((((x - 8)x + 28)x - 56)x + 70)x - 56)x + 28)x - 8)x + 1.$$

To evaluate $f(x)$ now requires seven multiplications and eight additions or subtractions. The necessity of using an exponential function to evaluate the polynomial has now been eliminated.

We end this section with the theorem that relates the Taylor series in Table 4.1 and the Taylor polynomials of Theorem 4.1.

Theorem 4.2 (Taylor Series). Assume that $f(x)$ is analytic on an interval (a, b) containing x_0 . Suppose that the Taylor polynomials (2) tend to a limit

$$(12) \quad S(x) = \lim_{N \rightarrow \infty} P_N(x) = \lim_{N \rightarrow \infty} \sum_{k=0}^N \frac{f^{(k)}(x_0)}{k!} (x - x_0)^k;$$

then $f(x)$ has the Taylor series expansion

$$(13) \quad f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!} (x - x_0)^k.$$

Proof. This follows directly from the definition of convergence of series in Section 1.1. The limit condition is often stated by saying that the error term must go to zero as N goes to infinity. Therefore, a necessary and sufficient condition for (13) to hold is that

$$(14) \quad \lim_{N \rightarrow \infty} E_N(x) = \lim_{N \rightarrow \infty} \frac{f^{(N+1)}(c)(x - x_0)^{N+1}}{(N + 1)!} = 0,$$

where c depends on N and x . •

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