Intro to Numerical Methods

APAM E4300 (1)

PROBLEM SET 9 – SOLUTIONS

(CHAPTER 11: INITIAL VALUE PROBLEM FOR ODEs)

Due: Monday, May 6, 2013

Problem 7 [5 points]:

7. Show that the classical fourth order Runge-Kutta method is stable by showing that when it is written in the form (11.3), the function \( \psi \) satisfies a Lipschitz condition.

For the classical fourth order Runge-Kutta method, \( \psi(t, y, h) = \frac{1}{6}[q_1(t, y, h) + 2q_2(t, y, h) + 2q_3(t, y, h) + q_4(t, y, h)] \), where

\[
q_1(t, y, h) = f(t, y), \quad q_2(t, y, h) = f(t + \frac{h}{2}, y + \frac{h}{2}q_1(t, y, h)),
\]

\[
q_3(t, y, h) = f(t + \frac{h}{2}, y + \frac{h}{2}q_2(t, y, h)), \quad q_4(t, y, h) = f(t + h, y + hq_3(t, y, h)).
\]

Assume that \( f \) satisfies the Lipschitz condition \( |f(t, y) - f(t, \tilde{y})| \leq L|y - \tilde{y}| \) for all \( y, \tilde{y}, \) and \( t \). Then

\[
|q_1(t, y, h) - q_1(t, \tilde{y}, h)| = |f(t, y) - f(t, \tilde{y})| \leq L|y - \tilde{y}|,
\]

\[
|q_2(t, y, h) - q_2(t, \tilde{y}, h)| = \left| f(t + \frac{h}{2}, y + \frac{h}{2}q_1(t, y, h)) - f(t + \frac{h}{2}, \tilde{y} + \frac{h}{2}q_1(t, \tilde{y}, h)) \right| 
\leq L \left| (y - \tilde{y}) + \frac{h}{2}(q_1(t, y, h) - q_1(t, \tilde{y}, h)) \right| 
\leq L|y - \tilde{y}| + L\frac{h}{2}|q_1(t, y, h) - q_1(t, \tilde{y}, h)| 
\leq L|y - \tilde{y}| + L\frac{h}{2}L|y - \tilde{y}| = \left( L + \frac{h}{2}L^2 \right)|y - \tilde{y}|.
\]

\[
|q_3(t, y, h) - q_3(t, \tilde{y}, h)| = \left| f(t + \frac{h}{2}, y + \frac{h}{2}q_2(t, y, h)) - f(t + \frac{h}{2}, \tilde{y} + \frac{h}{2}q_2(t, \tilde{y}, h)) \right| 
\leq L \left| (y - \tilde{y}) + \frac{h}{2}(q_2(t, y, h) - q_2(t, \tilde{y}, h)) \right| 
\leq L|y - \tilde{y}| + L\frac{h}{2}|q_2(t, y, h) - q_2(t, \tilde{y}, h)| 
\leq L|y - \tilde{y}| + L\frac{h}{2}\left( L + \frac{h}{2}L^2 \right)|y - \tilde{y}| 
= \left( L + \frac{h}{2}L^2 + \frac{h^2}{4}L^3 \right)|y - \tilde{y}|.
\]
Problem 8 [5 points]:

8. Continuing the love saga from Section ??, Juliet’s emotional swings lead to many sleepless nights, which consequently dampens her emotions. Mathematically, the pair’s love can now be expressed as

\[
\begin{align*}
\frac{dx}{dt} &= -0.2y \\
\frac{dy}{dt} &= 0.8x - 0.1y.
\end{align*}
\]

Suppose this state of the romance begins again when Romeo is smitten with Juliet (\(x(0) = 2\)) and Juliet is indifferent (\(y(0) = 0\)).

(a) Explain how the change in (??) that produces the equations above reflects Juliet’s dampened emotions.

Instead of Juliet’s love being just positively proportional to Romeo’s love, it is now negatively proportional to itself, meaning that the more Juliet loves Romeo the more her feelings start to get in the way and decrease her love.

(b) As in Section ??, use ode45 to produce 3 graphs, like those in Figure ??, showing Romeo and Juliet’s love for \(0 \leq t \leq 60\).
The following MATLAB code produces the plot below:

```matlab
[T,V] = ode45(@love,[0,60],[2,0]);

% Plot love of Romeo and Juliet vs t
subplot(1,3,1)
plot(T,V(:,1),'--', T,V(:,2),':--');
h = legend('Romeo','Juliet',2); axis tight
xlabel('t')

% Plot love of Romeo vs love of Juliet
subplot(1,3,2)
plot(V(:,1),V(:,2)), xlabel('x'), ylabel('y')
axis equal tight

% 3D plot of love of Romeo and love of Juliet vs t
subplot(1,3,3)
plot3(V(:,1),V(:,2),T)
xlabel('x'), ylabel('y'), zlabel('t')
axis tight

The file love.m contains:

function dvdt = love(t,v)
dvdt = zeros(2,1);
dvdt(1) = -0.2*v(2);
dvdt(2) = 0.8*v(1) - 0.1*v(2);
```

(c) From your graphs, describe how the change in Juliet described in this exercise will affect the relationship and its eventual outcome?

In this case, their love fizzles.