Intro to Numerical Methods
APAM E4300 (1)

Problem Set 3 – Solutions
(Single Nonlinear Equation in One Unknown)
Due: Monday, March 11, 2013

Problem 1 [4 points]:
The function $\varphi(x) = \frac{1}{2}(-x^2 + x + 2)$ has a fixed point at $x = 1$. Starting with $x_0 = 0.5$, we use $x_{n+1} = \varphi(x_n)$ to obtain the sequence $\{x_n\} = \{0.5; 1.1250; 0.9297; 1.0327; 0.9831; \ldots\}$. Describe the behavior of the sequence. (If it converges, how does it converge? If it diverges, how does it diverge?)

Solution:

$\varphi'(x) = \frac{1}{2}(-2x + 1)$. For $x \in (-\frac{1}{3}, \frac{3}{2})$, we have $|\varphi'(x)| < 1$, so in any interval about 1 of the form $[\frac{1}{2} + \delta, \frac{3}{2} - \delta]$ for some $\delta > 0$, $\varphi$ is a contraction. Now, $x_0$ is not quite in such an interval, but $x_1 = 1.1250$ is and consequently all subsequent iterates will be. Therefore the method converges to the unique fixed point in $[0.875, 1.125]$. Since $|\varphi'(1)| = \frac{1}{2}$, we expect linear convergence with the error being reduced by about a factor of 2 at each step. The convergence is oscillatory with iterates alternating between being less than 1 and being greater than 1.

Problem 2 [6 points]:
Recall that the Taylor series expansion of a function $f(x)$ about a point $x_0$ is

$$f(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{1}{2!}f''(x_0)(x-x_0)^2 + \frac{1}{3!}f'''(x_0)(x-x_0)^3 + \cdots$$

From this we can define a sequence of polynomials of increasing degree that approximate the function near the point $x_0$. The Taylor polynomial of degree $n$ is

$$P_n(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{1}{2}f''(x_0)(x-x_0)^2 + \frac{1}{6}f'''(x_0)(x-x_0)^3 + \cdots + \frac{1}{n!}f^{(n)}(x_0)(x-x_0)^n,$$

where $f^{(n)}(x_0)$ means the $n$th derivative of $f$ evaluated at $x_0$. 
The figure shows the function $f(x) = e^x$ and the first four Taylor polynomials from the expansion about the point $x_0 = 0$. Since all derivatives of $f(x) = e^x$ are again just $e^x$, the $n$th order Taylor polynomial from a series expansion about a general point $x_0$ is

$$P_n(x) = e^{x_0} + e^{x_0}(x - x_0) + \frac{1}{2} e^{x_0} (x - x_0)^2 + \frac{1}{6} e^{x_0} (x - x_0)^3 + \cdots + \frac{1}{n!} e^{x_0} (x - x_0)^n.$$ 

For $x_0 = 0$, we find

$$P_0(x) = 1,$$
$$P_1(x) = 1 + x,$$
$$P_2(x) = 1 + x + \frac{1}{2} x^2,$$
$$P_3(x) = 1 + x + \frac{1}{2} x^2 + \frac{1}{6} x^3,$$

and these are the functions plotted in the figure, along with $f(x)$.

Produce a similar set of four plots for the Taylor polynomials arising from the Taylor series expansion of $f(x) = e^{1-x^2}$ about the point $x_0 = 1$. You can start with the m-file (plotTaylor.m) used to create the figure above, which can be found on the web page for this class. Choose the x values and axis parameters so that the plots are over a reasonable range of values to exhibit the functions well. The polynomials should give good approximations to the function near the point $x_0 = 1$.

**Solution:**

First compute the Taylor polynomials for $e^{1-x^2}$ about $x_0 = 1$.

$$f(x) = e^{1-x^2}, \quad f(1) = 1$$
$$f'(x) = -2xe^{1-x^2}, \quad f'(1) = -2$$
$$f''(x) = (4x^2 - 2)e^{1-x^2}, \quad f''(1) = 2$$
$$f'''(x) = (-8x^2 + 12x)e^{1-x^2}, \quad f'''(1) = 4.$$
Therefore the first four terms in the Taylor series are:

\[ 1 - 2(x - 1) + (x - 1)^2 + \frac{2(x - 1)^3}{3}. \]

The following code produces the plot below:

```matlab
% Plot the first four Taylor polynomials for exp(1-x^2)
% about x_0 = 1.
x = [-2:.01:3]; % Set x values
fx = exp(1-x.^2);
p0 = ones(size(x)); % Compute Taylor polynomials
p1 = p0 - 2*(x-1);
p2 = p1 + (x-1).^2;
p3 = p2 + 2*(x-1).^3/3;

subplot(2,2,1) % Plot results
plot(x,fx,'--', x,p0,'-');
legend('f(x)','P_0(x)')
title('plot of P_0(x) and f(x)')
subplot(2,2,2)
plot(x,fx,'--', x,p1,'-');
legend('f(x)','P_1(x)')
title('plot of P_1(x) and f(x)')
subplot(2,2,3)
plot(x,fx,'--', x,p2,'-');
legend('f(x)','P_2(x)')
title('plot of P_2(x) and f(x)')
subplot(2,2,4)
plot(x,fx,'--', x,p3,'-');
legend('f(x)','P_3(x)')
title('plot of P_3(x) and f(x)')```