Intro to Numerical Methods
APAM E4300 (1)

PROBLEM SET 3 (SINGLE NONLINEAR EQUATION IN ONE UNKNOWN)

Due: Monday, March 11, 2013

Problem 1 [4 points]:
The function \( \phi(x) = \frac{1}{2}(-x^2 + x + 2) \) has a fixed point at \( x = 1 \). Starting with \( x_0 = 0.5 \), we use \( x_{n+1} = \phi(x_n) \) to obtain the sequence \( \{x_n\} = \{0.5; 1.1250; 0.9297; 1.0327; 0.9831; \ldots \} \). Describe the behavior of the sequence. (If it converges, how does it converge? If it diverges, how does it diverge?)

Problem 2 [6 points]:
Recall that the Taylor series expansion of a function \( f(x) \) about a point \( x_0 \) is
\[
f(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{1}{2!} f''(x_0)(x-x_0)^2 + \frac{1}{3!} f'''(x_0)(x-x_0)^3 + \cdots
\]
From this we can define a sequence of polynomials of increasing degree that approximate the function near the point \( x_0 \). The Taylor polynomial of degree \( n \) is
\[
P_n(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{1}{2} f''(x_0)(x-x_0)^2 + \frac{1}{6} f'''(x_0)(x-x_0)^3 + \cdots + \frac{1}{n!} f^{(n)}(x_0)(x-x_0)^n,
\]
where \( f^{(n)}(x_0) \) means the \( n \)th derivative of \( f \) evaluated at \( x_0 \).
The figure shows the function $f(x) = e^x$ and the first four Taylor polynomials from the expansion about the point $x_0 = 0$. Since all derivatives of $f(x) = e^x$ are again just $e^x$, the $n$th order Taylor polynomial from a series expansion about a general point $x_0$ is

$$P_n(x) = e^{x_0} + e^{x_0}(x-x_0) + \frac{1}{2!}e^{x_0}(x-x_0)^2 + \frac{1}{6}e^{x_0}(x-x_0)^3 + \cdots + \frac{1}{n!}e^{x_0}(x-x_0)^n.$$ 

For $x_0 = 0$, we find

$$P_0(x) = 1,$$

$$P_1(x) = 1 + x,$$

$$P_2(x) = 1 + x + \frac{1}{2}x^2,$$

$$P_3(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3,$$

and these are the functions plotted in the figure, along with $f(x)$.

Produce a similar set of four plots for the Taylor polynomials arising from the Taylor series expansion of $f(x) = e^{1-x^2}$ about the point $x_0 = 1$. You can start with the m-file (plotTaylor.m) used to create the figure above, which can be found on the web page for this class. Choose the x values and axis parameters so that the plots are over a reasonable range of values to exhibit the functions well. The polynomials should give good approximations to the function near the point $x_0 = 1$. 