Problem 1: (10 Points)
Determine whether each statement is true or false. If it is true, explain why. If it is false, explain why or give an example that disproves the statement.

(a) If \( f \) is differentiable at \( a \), then \( f \) is continuous at \( a \)
(b) If \( f \) is continuous at \( a \), then \( f \) is differentiable at \( a \)

Answer: 5 pts for each part (3 pts for correct answer, 2 pts for explanation).

(a) See Theorem 4 in Section 2.8, page 158 of the textbook.
(b) False. See the note after Theorem 4 in Section 2.8.

Problem 2: (10 Points)
Each limit represents the derivative of some function \( f \) at some number \( a \). State such an \( f \) and \( a \) in each case.

(a) \( \lim_{x \to 5} \left( \frac{2^x - 32}{x - 5} \right) \)
(b) \( \lim_{x \to \pi/4} \left( \frac{\tan x - 1}{x - \pi/4} \right) \)
(c) \( \lim_{h \to 0} \left( \frac{\cos(\pi + h) + 1}{h} \right) \)
(d) \( \lim_{t \to 1} \left( \frac{t^4 + t - 2}{t - 1} \right) \)

Answer: 2.5 pts for each part (1.5 pts for correct function, 1 pt for correct number.)
Problem 3: (10 Points)
Find the equations of the tangent lines to the curve
\[ y = \frac{2}{1-3x} \]
at the points with x-coordinates 0 and -1.
Answer: 4 pts for correct derivative, 3 pts for each correct tangent line.

Problem 4: (10 Points)
Prove that 
\[ \left(\frac{\tan^{-1}(x)}{x}\right)' = \frac{1}{1+x^2} \equiv \tan(y) = x \text{ and } -\frac{\pi}{2} < y < \frac{\pi}{2} \]
[Hint 1: \( \tan^{-1}(x) = y \iff \tan(y) = x \) and \(-\frac{\pi}{2} < y < \frac{\pi}{2}\)]
[Hint 2: \( 1 + \tan^2 y = \sec^2 y \)]
Answer: 5 pts for setting up problem correctly, 5 pts for correct solution.

See Section 3.5, page 214 of the textbook.

[4 Bonus Points] Compute the derivative of the function \( y = \tan^{-1}\sqrt{x} \)
Answer: 2 pts for setting up problem correctly, 2 pts for correct solution.

\[ y = \tan^{-1}\sqrt{x} \Rightarrow y' = \frac{1}{1 + \left(\sqrt{x}\right)^2} \cdot \frac{d}{dx} \left(\sqrt{x}\right) = \frac{1}{1 + x} \left(\frac{1}{2}x^{-1/2}\right) = \frac{1}{2\sqrt{x}(1 + x)} \]

Problem 5: (20 Points)
Compute the derivatives of the following functions:
1. \( f(x) = \sqrt{x} \cdot \sin x \)
2. \( f(x) = \frac{1-xe^{-x}}{x + e^x} \)
3. \( y = x \cdot \sin \left(\frac{1}{x}\right) \)
4. \( y = (\sqrt{x})^x \)

Answer: 4 pts for parts 1. and 3. 6 pts for parts 2. and 4.

\[
f(x) = \sqrt{x} \sin x \quad \Rightarrow \quad f'(x) = \sqrt{x} \cos x + \sin x \left( \frac{1}{2} \sqrt{x}^{-1/2} \right) = \sqrt{x} \cos x + \frac{\sin x}{2 \sqrt{x}}
\]

\[
f(x) = \frac{1 - xe^x}{x + e^x} \quad \Rightarrow \quad f'(x) = \frac{(x + e^x)(-xe^x)' - (1 - xe^x)(1 + e^x)}{(x + e^x)^2}
\]

\[
\Rightarrow \quad f'(x) = \frac{(x + e^x)(-xe^x) - (1 + e^x)(1 - xe^x)}{(x + e^x)^2} = \frac{-x^2 e^x - xe^x - xe^x - e^x - 1 - e^x + xe^x + xe^x - 1}{(x + e^x)^2} = \frac{-x^2 e^x - e^x - e^x - 1}{(x + e^x)^2}
\]

\[
y = x \sin \frac{1}{x} \quad \Rightarrow \quad y' = \sin \frac{1}{x} + x \cos \frac{1}{x} \left( \frac{-1}{x^2} \right) = \sin \frac{1}{x} - \frac{1}{x} \cos \frac{1}{x}
\]

\[
y = x \sin \frac{1}{x} \quad \Rightarrow \quad \ln y = \ln \sqrt{x} \quad \Rightarrow \quad \ln y = x \ln x^{1/2} \quad \Rightarrow \quad \ln y = \frac{1}{2} \ln x \quad \Rightarrow \quad \frac{1}{2} y' = \frac{1}{2} x \cdot \frac{1}{x} + \ln x \cdot \frac{1}{2} \quad \Rightarrow \quad y' = \sqrt{x} \ln x \quad \Rightarrow \quad y' = \sqrt{x} \ln x
\]

**Problem 6: (10 Points)**

Use Implicit Differentiation to find \( y'' \) if \( x^6 + y^6 = 1 \).

Answer: 6 pts for setting up problem correctly, 4 pts for correct solution.

\[
x^6 + y^6 = 1 \quad \Rightarrow \quad 6x^5 + 6y^5y' = 0 \quad \Rightarrow \quad y' = -\frac{x^5}{y^5} \quad \Rightarrow \quad y'' = -\frac{y^5(5x^4) - x^5(5y^4)}{(y^5)^3} = -\frac{5x^4y^4}{y^{14}} = -\frac{5x^4}{y^{10}}
\]

**Problem 7: (10 Points)**

A paper cup has the shape of a cone with height 10 cm and radius 3 cm (at the top). If water is poured into the cup at a rate of 2 cm\(^3\)/s, how fast is the water level rising when the water is 5 cm deep?

Answer: 5 pts for setting up problem correctly, 5 pts for correct solution.

Given \( \frac{dV}{dt} = 2 \), find \( \frac{dh}{dt} \) when \( h = 5 \). \( V = \frac{1}{3} \pi r^2 h \) and, from similar triangles, \( \frac{r}{h} = \frac{3}{10} \Rightarrow V = \frac{\pi}{3} \left( \frac{3h}{10} \right)^2 h = \frac{3\pi}{100} h^3 \), so

\[
2 = \frac{dV}{dt} = \frac{9\pi}{100} h^2 \frac{dh}{dt} \quad \Rightarrow \quad \frac{dh}{dt} = \frac{200}{9\pi (5)^2} = \frac{200}{9\pi} \frac{8}{9\pi} \text{ cm/s}
\]

when \( h = 5 \).
Problem 8: (10 Points)
A curve passes through the point (0,5) and has the property that the slope of the curve at every point P is twice the y-coordinate of P. What is the equation of the curve? Explain.

Answer: 5 pts for setting up problem correctly, 5 pts for correct solution.

From the information given, we know that \( \frac{dy}{dx} = 2y \) \( \Rightarrow \) \( y = Ce^{2x} \) by Theorem 2. To calculate \( C \) we use the point (0, 5):

\[ 5 = Ce^{2(0)} \Rightarrow C = 5. \]
Thus, the equation of the curve is \( y = 5e^{2x} \).

Problem 9: (10 Points)
Determine whether each statement is true or false. If it is true, explain why. If it is false, explain why or give an example that disproves the statement.

1. If \( f'(c) = 0 \), then \( f \) has a local maximum or minimum at \( c \)
2. If \( f \) has an absolute minimum value at \( c \), then \( f'(c) = 0 \)
3. If \( f \) is continuous on \((a, b)\), then \( f \) attains an absolute maximum value \( f(c) \) and an absolute minimum value \( f(d) \) at some numbers \( c \) and \( d \) in \((a, b)\)

Answer: 3 [2+1] pts for parts (1) and (2). 4 [2+2] pts for part (3).

1. False. For example, take \( f(x) = x^3 \), then \( f'(x) = 3x^2 \) and \( f'(0) = 0 \), but \( f(0) = 0 \) is not a maximum or minimum; \((0, 0)\) is an inflection point.

2. False. For example, \( f(x) = |x| \) has an absolute minimum at 0, but \( f'(0) \) does not exist.

3. False. For example, \( f(x) = x \) is continuous on \((0, 1)\) but attains neither a maximum nor a minimum value on \((0, 1)\). Don’t confuse this with \( f \) being continuous on the closed interval \([a, b]\), which would make the statement true.