COLUMBIA UNIVERSITY
CALCULUS I (MATH S1101X(3))

2ND SAMPLE MIDTERM 2 – JULY 12, 2012

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Problem 1: (10 Points)
Use the definition of derivative to prove that 
\[
\frac{d}{dx} \left( x + \sqrt{x} \right) = 1 + \frac{1}{2\sqrt{x}}.
\]
Answer:
\[
f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{(x+h + \sqrt{x+h}) - (x + \sqrt{x})}{h} \\
= \lim_{h \to 0} \left( \frac{h + \sqrt{x+h} - \sqrt{x}}{h} \cdot \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}} \right) = \lim_{h \to 0} \left[ 1 + \frac{(x+h) - x}{h(\sqrt{x+h} + \sqrt{x})} \right] \\
= \lim_{h \to 0} \left( 1 + \frac{1}{\sqrt{x+h} + \sqrt{x}} \right) = 1 + \frac{1}{2\sqrt{x}}
\]

Problem 2: (10 Points)
Find the equations of the tangent line and normal line to the parabola 
\[ y = (1 + 2x)^2 \]
at the point P(1,9).
Answer:
\[ y = (1 + 2x)^2 = 1 + 4x + 4x^2 \Rightarrow y' = 4 + 8x \]
At (1,9), \( y' = 12 \) and an equation of the tangent line is \( y - 9 = 12(x - 1) \) or \( y = 12x - 3 \). The slope of the normal line is \( -\frac{1}{12} \) (the negative reciprocal of 12) and an equation of the normal line is \( y - 9 = -\frac{1}{12}(x - 1) \) or \( y = -\frac{1}{12}x + \frac{105}{12} \).

Problem 3: (10 Points)
Let
\[
f(x) = \begin{cases} 
4x^2 + 4x + 1 & \text{if } x \leq 1 \\
mx + b & \text{if } x > 1
\end{cases}
\]
Find the values of \( m \) and \( b \) that make \( f \) differentiable everywhere.
Answer:
From Problem 2, it is immediate that we must have \( m = 12 \), and \( b = -3 \).
Problem 4: (20 Points)

Compute the derivatives of the following functions:

Answer:

1. \( f(x) = (x^3 + x)e^x \)

By the Product Rule, \( f(x) = (x^2 + 2x)e^x \) \( \Rightarrow \)

\[
f'(x) = (x^2 + 2x)(e^x)' + e^x(x^2 + 2x)' = (x^2 + 2x)e^x + e^x(3x^2 + 2) \]

\[
= e^x(x^2 + 3x^2 + 2x + 2) \]

2. \( f(t) = \frac{2t}{4 + t^2} \)

\[
f(t) = \frac{2t}{4 + t^2} \quad \Rightarrow \quad f'(t) = \frac{(4 + t^2)(2) - (2t)(2t)}{(4 + t^2)^2} = \frac{8 - 4t^2}{(4 + t^2)^2} \]

3. \( f(x) = \frac{x^2 - 2\sqrt{x}}{x} \)

\[
y = \frac{x^2 - 2\sqrt{x}}{x} = x - 2x^{-1/2} \quad \Rightarrow \quad y' = 1 - 2(-\frac{1}{2})x^{-3/2} = 1 + \frac{1}{x\sqrt{x}} \]

4. \( f(x) = x^{\cos(x)} \) (Hint: Use logarithmic differentiation)

\[
y = x^{\cos(x)} \quad \Rightarrow \quad \ln y = \ln x^{\cos(x)} \quad \Rightarrow \quad \ln y = \cos x \ln x \quad \Rightarrow \quad \frac{1}{y} y' = \cos x \cdot \frac{1}{x} + \ln x \cdot (-\sin x) \quad \Rightarrow \quad y' = y \left( \frac{\cos x}{x} - \ln x \sin x \right) \]

\[
y' = x^{\cos(x)} \left( \frac{\cos x}{x} - \ln x \sin x \right) \quad \Rightarrow \quad y' = x^{\cos(x)} \left( \cos x - \ln x \sin x \right) \]

Problem 5: (10 Points)

A table of values for \( f, g, f', \) and \( g' \) is given.

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f(x) )</th>
<th>( g(x) )</th>
<th>( f'(x) )</th>
<th>( g'(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>3</td>
<td>2</td>
<td>4</td>
<td>6</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>8</td>
<td>5</td>
<td>7</td>
</tr>
<tr>
<td>3</td>
<td>7</td>
<td>2</td>
<td>7</td>
<td>9</td>
</tr>
</tbody>
</table>
(a) If \( h(x) = f(g(x)) \), find \( h'(1) \).

(b) If \( H(x) = g(f(x)) \), find \( H'(1) \).

**Answer:**

(a) \( h(x) = f(g(x)) \Rightarrow h'(x) = f'(g(x)) \cdot g'(x) \), so \( h'(1) = f'(g(1)) \cdot g'(1) = f'(2) \cdot 6 = 5 \cdot 6 = 30 \).

(b) \( H(x) = g(f(x)) \Rightarrow H'(x) = g'(f(x)) \cdot f'(x) \), so \( H'(1) = g'(f(1)) \cdot f'(1) = g'(3) \cdot 4 = 9 \cdot 4 = 36 \).

**Problem 6: (10 Points)**

Use implicit differentiation to find \( \frac{dy}{dx} \) when \( x \) and \( y \) are related by the (ellipse) equation \( x^2 + xy + y^2 = 3 \).

Find the equation of the tangent line to the graph of \( x^2 + xy + y^2 = 3 \) at the point \((1,1)\).

**Answer:**

\[
\begin{align*}
x^2 + xy + y^2 &= 3 \\
2x + x' + y + 1 + 2y' &= 0 \\
x' + 2yy' - 2x - y &= 0 \\
y'(x + 2y) &= 2x + y \\
y' &= \frac{2x - y}{x + 2y}.
\end{align*}
\]

When \( x = 1 \) and \( y = 1 \), we have \( y' = \frac{-2 - 1}{1 + 2} = \frac{-3}{3} = -1 \), so an equation of the tangent line is \( y - 1 = -1(x - 1) \) or \( y = -x + 2 \).

**Problem 7: (10 Points)**

A particle moves along the curve \( y = \sqrt{1 + x^3} \). As it reaches the point \( P(2,3) \), the \( y \)-coordinate is increasing at a rate of 4 cm/s. How fast is the \( x \)-coordinate of the point changing at that instant?

**Answer:**

\[
\begin{align*}
y &= \sqrt{1 + x^3} \Rightarrow \frac{dy}{dt} &= \frac{dy}{dx} \cdot \frac{dx}{dt} = \frac{1}{2}(1 + x^3)^{-1/2}(3x^2) \cdot \frac{dx}{dt} = \frac{3x^2}{2\sqrt{1 + x^3}} \cdot \frac{dx}{dt}.
\end{align*}
\]

With \( \frac{dy}{dt} = 4 \) when \( x = 2 \) and \( y = 3 \), we have \( 4 = \frac{3(4)}{2(3)} \Rightarrow \frac{dx}{dt} = 2 \) cm/s.

**Problem 8: (10 Points)**

(a) Find the linearization of the function \( f(x) = \ln(x) \) near \( a = 1 \).

(b) Explain why the approximation \( \ln(1.05) \approx 0.05 \) is reasonable.

**Answer:**

\[
\begin{align*}
y &= f(x) = \ln x \Rightarrow f'(x) = \frac{1}{x}, \text{ so } f(1) = 0 \text{ and } f'(1) = 1. \text{ The linear approximation of } f \text{ at } 1 \text{ is } \\
f(1) + f'(1)(x - 1) &= 0 + 1(x - 1) = x - 1. \text{ Now } f'(1.05) = \ln 1.05 \approx 1.05 - 1 = 0.05, \text{ so the approximation is reasonable.}
\end{align*}
\]
Problem 9: (10 Points)
Consider the function \( f(x) = x^4 - 2x^2 + 3 \) having domain \([-2, 3]\). Find any absolute maximum or absolute minimum values of \( f(x) \), and find the x-values at which they occur.

Answer:

The function \( f \) is a polynomial, which is continuous everywhere, hence is continuous on \([-2, 3]\).

\[
\begin{align*}
f(x) &= x^4 - 2x^2 + 3, \quad [-2, 3]. \\
f'(x) &= 4x^3 - 4x = 4x(x^2 - 1) = 4x(x + 1)(x - 1) = 0 \iff x = -1, 0, 1.
\end{align*}
\]

\[
\begin{align*}
f(-2) &= 11, \\
f(-1) &= 2, \\
f(0) &= 3, \\
f(1) &= 2, \\
f(3) &= 66.
\end{align*}
\]

So \( f(3) = 66 \) is the absolute maximum value and \( f(\pm 1) = 2 \) is the absolute minimum value.