Problem 1: (10 Points)

Use the definition of derivative to prove that \( \frac{d}{dx} \left( 3x^2 + 1 \right) = 6x \).

Answer:

Let \( f(x) = 3x^2 + 1 \). To show that

\[
\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = 6x
\]

\[
= \lim_{h \to 0} \frac{3(x+h)^2 + 1 - 3x^2 - 1}{h} = \lim_{h \to 0} \frac{3(x^2 + 2xh + h^2) + 1 - 3x^2 - 1}{h} = \lim_{h \to 0} \frac{6xh + 3h^2}{h} = \lim_{h \to 0} 6x + 3h = 6x.
\]

Problem 2: (10 Points)

Find the equation of the tangent line to the parabola \( y = 3x^2 + 1 \) at the point \( P(1,4) \).

Answer:

From problem 1, we have that \( f'(x) = 6x \). Hence, the slope of the tangent line to the parabola at the point \( P(1,4) \) is \( f'(1) = 6 \). Hence the equation of the tangent line is given by \( y-4 = 6(x-1) \), or \( y = 6x - 2 \).

Problem 3: (10 Points)

Let

\[
f(x) = \begin{cases} 
3x^2 + 1 & \text{if } x \leq 1 \\
mx + b & \text{if } x > 1
\end{cases}
\]

Find the values of \( m \) and \( b \) that make \( f \) differentiable everywhere.

Answer:

From Problem 2, it is immediate that we must have \( m = 6 \), and \( b = -2 \).
Problem 4: (20 Points)
Compute the derivatives of the following functions:

1. \( f(x) = \sin^2 x \)
2. \( f(x) = \cos(x^2) \)
3. \( f(x) = \sqrt{x^2 + 1} \)
4. \( f(x) = x^x \) (Hint: Use logarithmic differentiation)

Answer:
1. Let \( g(x) = u = \sin(x) \), and let \( h(u) = u^2 \). It is easy to check that \( f(x) = h(g(x)) \). Hence by the chain rule, we have
   \[
   f'(x) = h'(u) \cdot g'(x) = 2u \cdot \cos(x) = 2\sin(x) \cdot \cos(x)
   \]
2. Let \( g(x) = u = x^2 \), and let \( h(u) = \cos(u) \). It is easy to check that \( f(x) = h(g(x)) \). Hence by the chain rule, we have
   \[
   f'(x) = h'(u) \cdot g'(x) = -\sin(u) \cdot (2x) = -2x \cdot \sin(x^2)
   \]
3. Let \( g(x) = u = x^2+1 \), and let \( h(u) = \sqrt{u} \). It is easy to check that \( f(x) = h(g(x)) \). Hence by the chain rule, we have
   \[
   f'(x) = h'(u) \cdot g'(x) = \frac{1}{2\sqrt{u}} \cdot (2x) = \frac{x}{\sqrt{x^2 + 1}}
   \]
4. Let \( y = x^u \). Then \( \ln(y) = \ln(x^u) = x \ln(x) \). By differentiating both sides, one obtains:
   \[
   \frac{1}{y} \frac{dy}{dx} = \frac{d}{dx} \ln(y) = \frac{d}{dx} (x \cdot \ln(x)) = 1 \cdot \ln(x) + x \cdot \frac{1}{x} = \ln(x) + 1.
   \]
   Hence: \( f'(x) = \frac{dy}{dx} = y \cdot (\ln(x) + 1) = x^u \cdot (\ln(x) + 1) \)

Problem 5: (10 Points)
Find a 2nd degree polynomial \( P \) such that \( P(2)=5 \), \( P'(2)=3 \), and \( P''(2)=2 \).

Answer:
Let \( P(x) = ax^2 + bx + c \). Then \( P'(x) = 2ax + b \), and \( P''(x) = 2a \).
The condition \( P''(2) = 2a = 2 \) implies that \( a = 1 \).
The condition \( P'(2) = 4a + b = 3 \) implies that \( b = 3 - 4a = 3 - 4(1) = -1 \)
The condition \( P(2) = 4a + 2b + c = 5 \) implies that \( c = 5 - 4a - 2b = 5 - 4(1) - 2(-1) = 3 \).
Hence \( P(x) = x^2 - x + 3 \).

Problem 6: (10 Points)
Use implicit differentiation to find \( \frac{dy}{dx} \) when \( x \) and \( y \) are related by the equation \( x^2 y = 1 \). Find the equation of the tangent line to the graph of \( x^2 y = 1 \) at the point \((1,1)\).
Answer:

By differentiating both sides of the equation \( x^2 y = 1 \), and by applying the product rule, one has:

\[
\frac{d}{dx} (x^2 y) = 2xy + x^2 \frac{dy}{dx} = \frac{d}{dx} (1) = 0,
\]

which implies \( \frac{dy}{dx} = -\frac{2y}{x} \). Hence the slope of tangent line to the graph of \( x^2 y = 1 \) at the point (1,1) equals -2. Therefore the equation of the tangent line is \( y - 1 = -2 (x - 1) \) or, equivalently, \( y = -2x + 3 \).

Problem 7: (10 Points)

Two cars start moving from the same point. One travels south at 60 mi/h and the other travels west at 25 mi/h. At what rate is the distance between the cars increasing two hours later?

Answer:

Let \( x \) denote the distance of the first car (traveling west) from the start point, let \( y \) denote the distance of the other car from the start point, and let \( z \) denote the distance between the two cars. We want to find \( \frac{dz}{dt} \). We know that \( \frac{dx}{dt} = 25 \text{ mi/h} \), \( \frac{dx}{dt} = 60 \text{ mi/h} \), and that the equation \( x^2 + y^2 = z^2 \) holds true. By differentiating both sides of the latter equation, one has:

\[
\frac{dx^2}{dt} + \frac{dy^2}{dt} = \frac{dz^2}{dt}
\]

or equivalently:

\[
2x \frac{dx}{dt} + 2y \frac{dy}{dt} = 2z \frac{dz}{dt}
\]

After two hours, we have that \( x = 50 \) miles, \( y = 120 \) miles, and \( z = 130 \) miles. By plugging these values in the latter equation, one finds that \( \frac{dz}{dt} = 65 \text{ mi/h} \).

Problem 8: (10 Points)

Find the linearization of the function \( f(x) = x^3 \) near \( a=1 \).

Answer:

The linearization \( L(x) \) of the function \( f \) near \( a \) is given by the equation \( L(x) = f(a) + f'(a) (x-a) \). In this case, \( f(x) = x^3 \) and \( a=1 \). Hence, \( f(1) = 1 \), \( f'(x) = 3x^2 \), and \( f'(1) = 3 \). Therefore:

\[
L(x) = f(1) + f'(1) (x-1) = 1 + 3 (x-1) = 3x - 2.
\]

I.e., \( f(x) \cong 3x – 2 \) for \( x \) near 1.

Problem 9: (10 Points)

Consider the function \( f(x) = x^4 - x^2 - 3 \) having domain \([-3, 3]\). Find any absolute maximum or absolute minimum values of \( f(x) \), and find the \( x \)-values at which they occur.

Answer:

\( f \) is a continuous function on the closed interval \([-3,3]\); hence we know that the function \( f \) must attain absolute max and min values on the closed interval \([-3,3]\).

Since \( f'(x) = 4x^3 - 2x = 2x (2x^2 -1) \), the critical numbers of the function \( f \) are \( x = 0 \), \( x = \pm 1/\sqrt{2} \).

Furthermore \( f(0) = -3 \), and \( f(1/\sqrt{2}) = f(-1/\sqrt{2}) = -13/4 \).

Finally the value of the function \( f \) at the endpoint is \( f(-3) = f(3) = 69 \).

Hence, the absolute max value is 69 (at \( x = \pm 3 \)), and the absolute min value is \(-13/4\) (at \( x = \pm 1/\sqrt{2} \)).