Solutions to Midterm 2
Calculus 1, Section 012
November 10, 2009

1. Differentiate the following functions with respect to $x$. (10 points each part = 30 points)

(a) $f(x) = \left(\frac{x+1}{x-1}\right) \arctan x$

**Solution:** Use the product rule, then the quotient rule.

\[
f'(x) = \frac{d}{dx} \left(\frac{x+1}{x-1}\right) \arctan x + \left(\frac{x+1}{x-1}\right) \frac{1}{1+x^2}
\]
\[
= \frac{x-1-(x+1)}{(x-1)^2} \arctan x + \left(\frac{x+1}{x-1}\right) \frac{1}{1+x^2}
\]
\[
= \frac{-2}{(x-1)^2} \arctan x + \left(\frac{x+1}{x-1}\right) \frac{1}{1+x^2}
\]

(b) $f(x) = e^{5 \tan x \cos x}$

**Solution:** First simplify, then apply the chain rule. (You can also apply chain rule and product rule without simplifying first, but it’s harder that way.)

\[
f(x) = e^{5 \tan x \cos x} = e^{5 \sin x}
\]
\[
f'(x) = (5 \cos x) e^{5 \sin x}
\]

(c) $f(x) = \sqrt{x + \sqrt{x + \cos x}}$

**Solution:** Apply the chain rule repeatedly.

\[
f'(x) = \frac{1}{2} (x + \sqrt{x + \cos x})^{-\frac{1}{2}} \cdot \frac{d}{dx} (x + \sqrt{x + \cos x})
\]
\[
= \frac{1}{2} (x + \sqrt{x + \cos x})^{-\frac{1}{2}} \left(1 + \frac{1}{2} (x + \cos x)^{-\frac{1}{2}} \frac{d}{dx} (x + \cos x)\right)
\]
\[
= \frac{1}{2} (x + \sqrt{x + \cos x})^{-\frac{1}{2}} \left(1 + \frac{1}{2} (x + \cos x)^{-\frac{1}{2}} (1 - \sin x)\right)
\]
2. Differentiate the following functions with respect to $x$. (10 points each part = 20 points)

(a) $f(x) = \frac{(x^3 + 2)\sqrt{3x - 1}}{(x^2 + 4)(x + 1)}$

**Solution:** You could use chain, product, and quotient rules, but logarithmic differentiation makes this problem much easier.

$$\ln(f(x)) = \ln \left( \frac{(x^3 + 2)\sqrt{3x - 1}}{(x^2 + 4)(x + 1)} \right)$$

$$\ln(f(x)) = \ln(x^3 + 2) + \frac{1}{2} \ln(3x - 1) - \ln(x^2 + 4) - \ln(x + 1)$$

Taking derivatives of each side, and using the chain rule, we get:

$$\frac{f'(x)}{f(x)} = \frac{3x^2}{x^3 + 2} + \frac{3}{2(3x - 1)} - \frac{2x}{x^2 + 4} - \frac{1}{x + 1}$$

$$f'(x) = \frac{(x^3 + 2)\sqrt{3x - 1}}{(x^2 + 4)(x + 1)} \left( \frac{3x^2}{x^3 + 2} + \frac{3}{6x - 2} - \frac{2x}{x^2 + 4} - \frac{1}{x + 1} \right)$$

(b) $f(x) = (\tan x)^{\tan x}$

**Solution:** Use logarithmic differentiation.

$$\ln(f(x)) = \ln((\tan x)^{\tan x})$$

$$\ln(f(x)) = \tan x \ln(\tan x)$$

$$\frac{f'(x)}{f(x)} = \sec^2 x \ln(\tan x) + \tan x \frac{\sec^2 x}{\tan x}$$

$$f'(x) = \frac{f'(x)}{f(x)} \cdot f(x) = \sec^2 x \ln(\tan x) + \sec^2 x$$

$$f(x) = (\tan x)^{\tan x} \left( \sec^2 x \ln(\tan x) + \sec^2 x \right)$$
3. You and your friend start at the same point. You walk south at a rate of 2 mi/hr and your friend walks east at a rate of 4 mi/hr. How fast is the distance between you and your friend increasing after you have walked 4 miles? (20 points)

**Solution:** Define functions representing how far you have walked and how far your friend has walked and the distance between you.

\[ f(t) = \text{how far you have walked at time } t \]
\[ g(t) = \text{how far your friend has walked at time } t \]
\[ d(t) = \text{the distance between you and your friend at time } t \]

Their derivatives represent the speed you’re walking at time \( t \), the speed your friend is walking at time \( t \), and how fast the distance between you is increasing at time \( t \), respectively.

Since you’re going south and your friend is going east, your paths are making a right triangle. Use the Pythagorean theorem to get a relation between \( f \), \( g \), and \( d \).

\[ f(t)^2 + g(t)^2 = d(t)^2 \]  \hspace{1cm} (1)

Now differentiate this relationship, making sure to use the chain rule.

\[ 2f(t)f'(t) + 2g(t)g'(t) = 2d(t)d'(t), \]

which simplifies to

\[ f(t)f'(t) + g(t)g'(t) = d(t)d'(t). \]  \hspace{1cm} (2)

Now fill in the information you know:

- you walk south at a rate of 2 mi/hr \( \Rightarrow \) \( f'(t) = 2 \)
- your friend walks east at a rate of 4 mi/hr \( \Rightarrow \) \( g'(t) = 4 \)
- after you have walked 4 miles \( \Rightarrow \) \( f(t) = 4 \)

Since your friend is walking twice as fast as you are, you also know that your friend has gone 8 miles once you’ve gone 4 miles. So you know \( g(t) = 8 \). Putting all of this into equation 2 above, you have

\[ 4 \cdot 2 + 8 \cdot 4 = d(t)d'(t) \]

so \( 40 = d(t)d'(t) \)

The problem asks how fast that distance between you and your friend is increasing, so that’s \( d'(t) \). The missing piece now is \( d(t) \), which you can find by substituting what you know into equation 1:

\[ 4^2 + 8^2 = 80 = d(t)^2 \]

so \( d(t) = \sqrt{80} \).

Now you can finish solving the previous equation for \( d'(t) \):

\[ d'(t) = \frac{40}{\sqrt{80}} = \frac{10}{\sqrt{5}} \text{ mi/hr.} \]
4. (a) State the Extreme Value Theorem. (5 points)

**Solution:** If \( f \) is continuous on \([a, b]\), then \( f \) achieves an absolute maximum and an absolute minimum on \([a, b]\).

(b) Give an example that demonstrates why the premise “\( f \) is continuous” is necessary for the extreme value theorem to be true. Graphs and formulas are equally acceptable, but explain your example in either case. (10 points)

**Solution:** Many correct answers are possible. You need to write down a function \( f \) and a closed interval \([a, b]\) such that \( f \) is discontinuous at some point in \([a, b]\) and either \( f \) fails to achieve an absolute maximum on \([a, b]\) or \( f \) fails to achieve an absolute minimum on \([a, b]\). Here’s a possibility with a hole discontinuity where the absolute minimum would be: \( f(x) = \frac{(x-1)^3}{x-1} \) on \([0, 2]\). Here’s a possibility with a jump discontinuity where the absolute maximum would be:

\[
f(x) = \begin{cases} 
  x + 1 & \text{if } x < 0 \\
  -x & \text{if } x \geq 0
\end{cases}
\text{ on the interval } [-1, 1].
5. Throughout this problem, consider the equation \(4xy = x^3 - y^3 + 4y\).

(a) Find an equation for the tangent line to the graph of this equation at the point \((1,1)\). (12 points)

Solution: First use implicit differentiation (differentiate everything with respect to \(x\)) to find \(y'\), which will give the slope of tangent lines to this graph.

\[
4y + 4xy' = 3x^2 - 3y^2 y' + 4y'
\]

\[y' = \frac{3x^2 - 4y}{4x + 3y^2 - 4}\]

To find the tangent line at the point \((1,1)\), set \(x\) and \(y\) equal to 1 in the above equation. This gives \(y' = -\frac{1}{3}\). Then the tangent line is

\[y - 1 = -\frac{1}{3} (x - 1)\]

(b) Find at least one point at which this graph has a horizontal tangent line. (8 points)

Solution: The graph will have a horizontal tangent line if the numerator of the equation for \(y'\) is zero. This occurs when \(4y = 3x^2\). There is more than one choice of values for \(x\) and \(y\) that make this equation true (so there is more than one horizontal tangent line), but the easiest choice is \(x = y = 0\). So the graph has a horizontal tangent line at the origin.

(c) (5 points) Given what you found out in parts (a) and (b), which of the three graphs on the next page could be the graph of this equation? (5 points)

Solution: The graph on the left is correct. Based on the answer to part (b) of this question, you should be looking for a graph with a horizontal tangent line at the origin. That rules out the middle graph, which has a vertical tangent line at the origin. Based on the answer to part (a) of this problem, you should be looking for a graph whose slope at \((1,1)\) is negative. That rules out the graph on the right, which does not appear to go through \((1,1)\) at all, and anyway would has positive slope at any of the points near \((1,1)\) that it does go through.
6. The goal of this problem is to sketch the graph of the function

\[ f(x) = \frac{x^3 - 2}{3x}. \]

You may use the following facts without showing why they are true:

\[ f'(x) = \frac{2x^3 + 2}{3x^2}, \quad f''(x) = \frac{2x^3 - 4}{3x^3} \quad \text{and} \quad \sqrt[3]{2} \approx 1.26. \]

This problem is worth 40 points all together, 25 of which are for answering the questions that help you sketch the graph (parts (a)-(i)) and 15 for drawing a graph that matches your answers to parts (a)-(i).

(a) What is the domain of the function \( f \)? (1 point)

**Solution:** The domain is all real numbers except 0, so \((-\infty, 0) \cup (0, \infty)\).

(b) What are the \( x \)-intercepts and \( y \)-intercepts (if any) of \( f \)? (2 points)

**Solution:** There are no \( y \)-intercepts because 0 is not in the domain of \( f \). The \( x \)-intercepts occur when

\[ f(x) = \frac{x^3 - 2}{3x} = 0, \]

so at \( x = \sqrt[3]{2} \). The coordinate is \((\sqrt[3]{2}, 0)\).

(c) Find equations for any horizontal asymptotes of \( f \) or say why \( f \) has no horizontal asymptotes. (3 points)

**Solution:** Find horizontal asymptotes by looking at the limits as \( x \to \pm\infty \). Since the numerator is a higher degree polynomial than the denominator, the limits are going to be \( \infty \) or \(-\infty\). You can solve this informally by thinking about the sign of \( f \) as \( x \to \infty \) or \( x \to -\infty \), or you can use the method we learned earlier in the semester (dividing the numerator and denominator by the highest degree term). Either way, you should get

\[ \lim_{x \to -\infty} \frac{x^3 - 2}{3x} = \infty \quad \text{and} \quad \lim_{x \to \infty} \frac{x^3 - 2}{3x} = \infty \]

Therefore, \( f \) has no horizontal asymptotes.

(d) Find equations for any vertical asymptotes of \( f \) or say why \( f \) has no vertical asymptotes. (3 points)

**Solution:** The only place where \( f \) is discontinuous is as 0, so the only vertical asymptote is \( x = 0 \). You can check that it really is a vertical asymptote by taking the limit as \( x \to 0 \) from the left and from the right:

\[ \lim_{x \to 0^-} \frac{x^3 - 2}{3x} = \infty \quad \text{and} \quad \lim_{x \to 0^+} \frac{x^3 - 2}{3x} = -\infty \]

(e) What are the critical numbers of \( f \)? (3 points)

**Solution:** Critical numbers are places where the function is not differentiable or where its derivative is zero. It’s not defined at \( x = 0 \), so it’s not differentiable there. The derivative is zero when

\[ f'(x) = \frac{2x^3 + 2}{3x^2} = 0, \]

so when \( 2x^3 + 2 = 0 \), so when \( x^3 = -1 \), so when \( x = -1 \).

Therefore, the critical numbers are \( x = 0 \) and \( x = -1 \).
(f) On what intervals is \( f \) increasing and decreasing? (4 points)

**Solution:** Since the critical numbers are at \( x = 0 \) and \( x = -1 \), we check whether \( f' \) is positive or negative on the intervals \((-\infty, -1)\) and \((-1, 0)\) and \((0, \infty)\). You could test points, or you could reason as follows.

\((-\infty, -1)\) : numerator of \( f' \) is negative, denominator is positive \(\implies\) negative
\((-1, 0)\) : numerator of \( f' \) is positive, denominator is positive \(\implies\) positive
\((0, \infty)\) : numerator of \( f' \) is positive, denominator is positive \(\implies\) positive

Therefore, \( f \) is decreasing on \((-\infty, -1)\) and increasing everywhere else.

(g) Does \( f \) have any local maxima or minima? If so, what are their coordinates? (2 points)

**Solution:** Local extrema occur if \( f' \) changes sign around a critical point, so \( x = -1 \) is a local extremum. Since \( f' \) changes from negative to positive, it’s a local minimum. Since \( f(-1) = 1 \), the coordinates of the local minimum are \((-1, 1)\).

(h) On what intervals is \( f \) concave upward and concave downward? (5 points)

**Solution:** Look at the sign of \( f'' \) to answer this question. To make that easier, first find where \( f'' = 0 \).

\[
f''(x) = \frac{2x^3 - 4}{3x^3} = 0 \text{ if } 2x^3 - 4 = 0, \text{ so if } x^3 = 2, \text{ so if } x = \sqrt[3]{2}.
\]

Also, \( f'' \) is undefined at \( x = 0 \). Therefore, we check whether \( f'' \) is positive or negative on \((-\infty, 0)\) and on \((0, \sqrt[3]{2})\) and on \((\sqrt[3]{2}, \infty)\). You can check points or reason as follows.

\((-\infty, 0)\) : numerator of \( f'' \) is negative, denominator is negative \(\implies\) positive
\((0, \sqrt[3]{2})\) : numerator of \( f'' \) is negative, denominator is positive \(\implies\) negative
\((\sqrt[3]{2}, \infty)\) : numerator of \( f'' \) is positive, denominator is positive \(\implies\) positive

Therefore, \( f \) is concave down on \((0, \sqrt[3]{2})\) and concave up everywhere else.

(i) Does \( f \) have any inflection points? If so, what are their coordinates? (2 points)

**Solution:** An inflection point is a place where a function changes concavity, so when \( f'' \) changes sign. Therefore, \( f \) has an inflection point at \( x = \sqrt[3]{2} \). Since \( f(\sqrt[3]{2}) = 0 \), its coordinates are \((\sqrt[3]{2}, 0)\).
(j) Sketch a graph of $f$ that shows each of the features you identified in parts (a)-(i). You may use the grid provided below or sketch in the blank part of the page. (15 points)

**Solution:**

- **inflection point and x-intercept**
- **local min**
- **vertical asymptote**
Extra Credit

EC1) (10 points) Give an example that demonstrates why the following statement is false.

If \( f \) is continuous on the interval \((a, b)\), then \( f \) achieves an absolute maximum and an absolute minimum on \((a, b)\).

**Solution:** Again, lots of possible examples. The point here is that the extreme value theorem is only true on closed intervals, not on open intervals. The example you write down should be continuous on an open interval, but either not attain an absolute max or not attain an absolute min on that interval. One possible example: \( f(x) = x \) on any open interval you like. Another possibility: \( f(x) = 1/x \) on \((0, 3)\). Even though it’s not continuous at 0, it is continuous on the open interval \((0, 3)\). Since \( f \) goes to infinity as \( x \) goes to 0, this function definitely doesn’t attain an absolute maximum on \((0, 3)\).

EC2) (15 points) Prove the following.

If \( f(0) = 0 \) and \( f'(x) \geq 0 \) for all \( x > 0 \), then \( f(x) \geq 0 \) for all \( x \geq 0 \).

**Solution:** [You also had to assume \( f \) was continuous and differentiable – sorry.] The key here is to apply the mean value theorem. Let \( b \) be any number greater than 0. Then \( f \) is differentiable on \((0, b)\) and continuous on \([0, b]\) by assumption, so the mean value theorem applies. It says that there exists a number \( c \) in the interval \((0, b)\) such that

\[
f'(c) = \frac{f(b) - f(0)}{b - 0} = \frac{f(b)}{b}.
\]

(3)

Since we know that \( f'(x) \geq 0 \) for all \( x > 0 \), we know that \( f'(c) \geq 0 \). We also chose \( b > 0 \). Therefore, by equation (1), we must have \( f(b) \geq 0 \). Since this argument was for an arbitrary \( b > 0 \), we have shown that \( f(x) \geq 0 \) for any \( x > 0 \).