V1101.001 Spring 2010
Calculus I - Section 1
Second Midterm

Name:______________________________

Instructions: You have 75 minutes to complete the exam. Calculators, notes and textbooks are not allowed. Provide the answers in the simplest possible form that does not require calculator use (e.g. expressions like $\sqrt{7}$ are fine). Show all of your work. If you only give a numerical answer you will receive no credit. Partial credit will be given for partial solutions. Write your solutions in the space below the questions. If you need more space, use the back of the page. **Do not forget to write your name in the space above.**

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1. Compute the derivatives of the following functions.
(a) \( f(x) = \frac{4x^3 + \sin(x)}{e^x + \cos(x)} \)
(b) \( f(x) = \arctan(\cos(x^2)) \)

Solution: (a) We first apply the quotient rule and get:
\[
f'(x) = \frac{(4x^3 + \sin(x))' \cdot (e^x + \cos(x)) - (4x^3 + \sin(x)) \cdot (e^x + \cos(x))'}{(e^x + \cos(x))^2}
\]
The derivatives in the numerator are:
\[(4x^3 + \sin(x))' = 12x^2 + \cos(x)\]
\[(e^x + \cos(x))' = e^x - \sin(x)\]
Hence we get:
\[
f'(x) = \frac{(12x^2 + \cos(x)) \cdot (e^x - \sin(x)) - (4x^3 + \sin(x)) \cdot (e^x - \sin(x))}{(e^x + \cos(x))^2}
\]
(b) Note that \( f(x) = g(h(x)) \), where \( g(x) = \arctan(x) \) and \( h(x) = \cos(x^2) \). Now \( g'(x) = \frac{1}{1 + x^2} \), hence by the chain rule we have:
\[
f'(x) = g'(h(x)) \cdot h'(x) = \frac{1}{1 + \cos(x^2)^2} \cdot (\cos(x^2))'
\]
To find \((\cos(x^2))'\), we need to apply the chain rule again. That is, note that \( \cos(x^2) = g(h(x)) \), where \( g(x) = \cos(x) \) and \( h(x) = x^2 \). Since \( g'(x) = -\sin(x) \) and \( h'(x) = 2x \), the chain rule gives:
\[(\cos(x^2))' = g'(h(x)) \cdot h'(x) = -\sin(x^2) \cdot 2x\]
Substituting in the expression for \( f'(x) \), we finally find:
\[
f'(x) = -\frac{1}{1 + \cos(x^2)^2} \sin(x^2) \cdot 2x
\]
2. Using linear approximation, find the approximate value of \( \sqrt{0.996} \).

Solution: Consider the function \( f(x) = \sqrt{x} \). We have \( f'(x) = \frac{1}{4}x^{-\frac{3}{2}} \). Linear approximation asserts that for \( x \) very close to 1, we have:

\[
f(x) \approx f(1) + f'(1)(x - 1)
\]

In our case \( f(1) = 1 \) and \( f'(1) = \frac{1}{4} \) and linear approximation gives:

\[
\sqrt{0.996} = f(0.996) \approx f(1) + f'(1)(0.996 - 1) = 1 + \frac{1}{4}(0.996 - 1) = 0.999
\]
For $x > 0$, consider the function

$$f(x) = 2x + \frac{1}{x^2}$$

(a) Find the critical points of $f$.
(b) Determine the global maximum and the global minimum of $f$ on the interval $[\frac{1}{2}, 2]$.

Solution: (a) The critical points of $f$ are the numbers $c$ such that $f'(c) = 0$ or $f'(c)$ does not exist. Since the function $f(x)$ is given by a sum of rational functions, it is differentiable everywhere on its domain. Hence the critical points are the points $x > 0$ such that $f'(x) = 0$. We compute $f'(x)$:

$$f'(x) = 2 - \frac{2}{x^3} = \frac{2x^3 - 2}{x^3}$$

Thus $f'(x) = 0 \iff 2x^3 - 2 = 0 \iff x = 1$ and the only critical point is $x = 1$.

(b) Since we already know the critical points of $f$, we only need to evaluate $f$ at those points and at the extreme points of the interval: $x = \frac{1}{2}, x = 2$:

$$f\left(\frac{1}{2}\right) = 1 + 4 = 5$$
$$f(1) = 2 + 1 = 3$$
$$f(2) = 4 + \frac{1}{4} = \frac{17}{4}$$

Thus the global maximum is attained at $x = \frac{1}{2}$ with value $f\left(\frac{1}{2}\right) = 5$ and the global minimum is attained at $x = 1$ with value $f(1) = 3$. 

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4. (a) State the Mean Value Theorem.
(b) If a function is continuous at $x = a$, is it necessarily differentiable at $x = a$? Justify your answer by referring to theorems or giving a counterexample.

Solution: (a) If $f$ is continuous on $[a, b]$ and differentiable on $(a, b)$, then there exists $c$ in $(a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

(b) The answer is no. An example is provided by the function $f(x) = |x|$. This function is continuous at 0 (in fact, it is continuous everywhere), but it is not differentiable at 0.
5. A particle is moving along the curve $x^2 - 3xy + y^2 = -1$. As it passes through the point $(x = 1, y = 1)$, its velocity in the $x$-direction is $3 \text{m/s}$. What is its velocity in the $y$-direction?

Solution: We consider $x$ and $y$ as functions of the time $t$ and we apply implicit differentiation to the equation $x^2 - 3xy + y^2 = -1$ (we denote differentiation with respect to $t$ by the superscript $'$):

$$ (x^2 - 3xy + y^2)' = (-1)' $$

Obviously $(-1)' = 0$. We also have:

$$ (x^2 - 3xy + y^2)' = 2xx' - 3x'y - 3xy' + 2yy' $$

Hence:

$$ 2xx' - 3x'y - 3xy' + 2yy' = 0 \iff (3x - 2y)y' = (2x - 3y)x' $$

Substituting $x = 1$, $y = 1$, $x' = 3$ we get $3x - 2y = 1$, $2x - 3y = -1$ and hence:

$$ y' = -x' = -3 \text{m/s} $$
6. Consider the function $f(x) = x^3 - 9x^2 + 24x$.

(a) Find all the critical points of $f$. Then find all its local maxima and minima on $(-\infty, +\infty)$.

(b) Describe the intervals where $f$ is increasing and decreasing.

(c) Find all the inflection points of $f$ on $(-\infty, +\infty)$. Describe the intervals where $f$ is concave up and concave down.

(d) Using the results of (a), (b) and (c), sketch the graph of $f(x)$.

Solution: (a) The function $f$ is a polynomial, hence it is differentiable everywhere and the critical points are the solutions to the equation $f'(x) = 0$. We compute the first derivative of $f(x)$:

$$f'(x) = 3x^2 - 18x + 24 = 3(x^2 - 6x + 8)$$

Thus:

$$f'(x) = 0 \iff x^2 - 6x + 8 = 0 \iff x = 2, \ x = 4$$

To find the local maxima and minima, we use the second derivative test. We have:

$$f''(x) = 6x - 18$$

Hence:

$$f'(2) = 0, \ f''(2) = 12 - 18 = -6 < 0 \Rightarrow x = 2 \text{ is a local maximum.}$$

$$f'(4) = 0, \ f''(4) = 24 - 18 = 6 > 0 \Rightarrow x = 4 \text{ is a local minimum.}$$

(b) Since $f'(x) = 3(x - 2)(x - 4)$, we find that $f'(x)$ is positive (and hence $f(x)$ is increasing) for $x \in (-\infty, 2) \cup (4, +\infty)$ and $f'(x)$ is negative (hence $f(x)$ is decreasing) for $x \in (2, 4)$.

(c) To find the points of inflection, we have to set $f''(x) = 0$. We get:

$$f''(x) = 6x - 18 = 0 \iff x = 3$$

Since $f''(x) > 0$ for $x > 3$, the function is concave upward when $x > 3$. Since $f''(x) < 0$ for $x < 3$, the function is concave downward when $x < 3$. Since the function changes from concave downward to concave upward at $x = \frac{1}{2}$, this point is an inflection point.