1. Give an example of a function with domain and range $\mathbb{R}$ that does not have an inverse. Your answer can be a formula or a graph. Explain why the function you chose does not have an inverse. (10 points)

Solution: Any function that fails the horizontal line test (and has the proper range and domain) would work here. Equivalently, any function for which two different inputs can give the same output does not have an inverse. One example is the cubic function shown below, but there are many more.

2. Let $f(x) = x^2 - 1$.

(a) Is $f$ even, odd, or neither? Justify your answer. (5 points)

Solution: The function $f$ is even because $f(-x) = (-x)^2 - 1 = x^2 - 1 = f(x)$. An alternative justification is to draw the graph of $f$ (shown below) and note that it is symmetric across the vertical axis.

(b) Let $g(x)$ be the function whose graph is translated 3 units to the right from the graph of $f(x)$. Give the formula for $g(x)$. (5 points)

Solution: To translate the graph of $f$ to the right, we find $f(x - 3)$. Therefore, $g(x) = f(x - 3) = (x - 3)^2 - 1 = x^2 - 6x + 8$.

(c) On the same diagram, draw the graph of $g(x)$, a secant line for $g(x)$ that passes through the point $(3, g(3))$, and the tangent line to $g$ at $x = 3$. Make sure to label your graph clearly and show detail. (10 points)

Solution: See the graph below. You could have drawn other secant lines, just so long as they passed through $(3, -1)$ and one other point on the graph of $g$. 


3. Calculate each limit or explain why it does not exist. (10 points each part = 30 points)

(a) \( \lim_{x \to 0} (x^2 + \sqrt[3]{x}) \sin \left( x + e^{\frac{x}{2}} \right) \)

**Solution:** You cannot evaluate this limit by substitution or limit laws since \( e^{\frac{x}{2}} \) goes to \( \infty \) as \( x \to 0 \) and \( \lim_{t \to \infty} \sin t \) does not exist. Instead, use the squeeze theorem.

Since the domain of sine is \([-1, 1]\), we can bound \(-1 \leq \sin \left( x + e^{\frac{x}{2}} \right) \leq 1\). Then

\[-(x^2 + \sqrt[3]{x}) \leq (x^2 + \sqrt[3]{x}) \sin \left( x + e^{\frac{x}{2}} \right) \leq x^2 + \sqrt[3]{x}\]

The limits on either side can be evaluated by direct substitution because polynomials and root functions are continuous:

\( \lim_{x \to 0} -(x^2 + \sqrt[3]{x}) = 0 \) and \( \lim_{x \to 0} (x^2 + \sqrt[3]{x}) = 0 \).

Therefore, by the squeeze theorem, \( \lim_{x \to 0} (x^2 + \sqrt[3]{x}) \sin \left( x + e^{\frac{x}{2}} \right) = 0 \).

(b) \( \lim_{x \to 2} \frac{x^2 - 4}{|x - 2|} \)

**Solution:** It’s tempting to cancel, but be careful with the absolute value! Split this limit into a left and right limit. Since \( x > 2 \) implies \( x - 2 > 0 \), we can replace \( |x - 2| \) with \( x - 2 \) in the right limit:

\[ \lim_{x \to 2^+} \frac{x^2 - 4}{x - 2} = \lim_{x \to 2^+} \frac{x^2 - 4}{x - 2} \]

\[ = \lim_{x \to 2^+} \frac{(x - 2)(x + 2)}{x - 2} \]

\[ = \lim_{x \to 2^+} x + 2 \]

\[ = 4. \]

For the left limit, we’re only dealing with \( x < 2 \), which means \( x - 2 < 0 \), so we can replace \( |x - 2| \) with \(- (x - 2)\):

\[ \lim_{x \to 2^-} \frac{x^2 - 4}{|x - 2|} = \lim_{x \to 2^-} \frac{x^2 - 4}{-(x - 2)} \]

\[ = \lim_{x \to 2^-} \frac{(x - 2)(x + 2)}{-(x - 2)} \]

\[ = \lim_{x \to 2^-} -(x + 2) \]

\[ = -4. \]

Since the left and right limits don’t agree, this limit doesn’t exist.
\[ \lim_{x \to \infty} \frac{x^5 - 3x + 2}{5 - 4x^4} \]

**Solution:** Divide the numerator and denominator of this limit by \( x^4 \) to get a limit you can calculate using the limit laws.

\[
\lim_{x \to \infty} \frac{x^5 - 3x + 2}{5 - 4x^4} = \lim_{x \to \infty} \frac{x^5 - 3x + 2}{x^4} = \lim_{x \to \infty} \frac{\frac{x^5}{x^4} - \frac{3x}{x^4} + \frac{2}{x^4}}{\frac{5}{x^4} - \frac{4}{x^4}} = \lim_{x \to \infty} \frac{x - \frac{3}{x^3} + \frac{2}{x^4}}{\frac{5}{x^4} - \frac{4}{x^4}} = \lim_{x \to \infty} \frac{x - 0 + 0}{4} = -\infty.
\]

4. (a) State the intermediate value theorem. (5 points)

**Solution:** Let \( f \) be a continuous on the interval \([a, b] \) with \( f(a) \neq f(b) \). For any number \( N \) between \( f(a) \) and \( f(b) \), there is a number \( c \) in \((a, b)\) such that \( f(c) = N \).

(b) Use the intermediate value theorem to show that \( p(x) = x^3 + 5x^2 + x - 2 \) has a root. (A root is a number \( a \) such that \( p(a) = 0 \).) Be sure to say why the theorem applies. (10 points)

**Solution:** The intermediate value theorem applies to \( p \) on any interval because polynomials are continuous everywhere on their domains, which means everywhere on \( \mathbb{R} \). To show that \( p \) has a root, we just need to find some interval \([a, b]\) such that \( p(a) \) is positive and \( p(b) \) is negative or vice versa. Many intervals work, but a convenient choice is \((-1, 0)\). We check that \( p(-1) = 1 \) and \( p(0) = -2 \). Since 0 is between \( p(-1) \) and \( p(0) \), the intermediate value theorem tells us that there is some \( c \) between -1 and 0 such that \( p(c) = 0 \).

**Extra Credit (5 points):** Show that \( p \) has two distinct roots.

**Solution:** To show that \( p \) has two distinct roots, we can use the intermediate value theorem to find roots of \( p \) in two non-overlapping intervals. For example, we have already found a root in \((-1, 0)\). We could also check the interval \([0, 1]\). The polynomial \( p \) is continuous on this interval too and \( p(0) = -2 \) while \( p(1) = 5 \). Since 0 is between \( p(-1) \) and \( p(0) \), the intermediate value theorem guarantees that there is some \( c' \) in the interval \((0, 1)\) such that \( p(c') = 0 \). So we have found \( c \) and \( c' \), which are both roots of \( p \) and which must be distinct because they are in disjoint intervals.

(c) Let \( f(x) = \frac{x^2 - 9}{x - 3} \). Then \( f(0) = 3 \) and \( f(4) = 7 \). Why can’t the intermediate value theorem be used to show that \( f(c) = 6 \) for some \( c \)? (10 points)

**Solution:** The intermediate value theorem can only be applied to a function on an interval where it is continuous. The function \( f \) is not continuous on the interval \([0, 4]\). It has a removable discontinuity (a hole in its graph) at \( x = 3 \). You could show this by graphing \( f \). You could also argue that \( \lim_{x \to 3} f(x) = 6 \), but \( f(3) \) does not make sense because 3 is not in the domain of \( f \). Therefore, \( f \) is not continuous at \( x = 3 \).
5. Let \( f(x) = x^2 - 5x \). Use the limit definition of the derivative to calculate \( f'(x) \). (15 points)

**Solution:** The limit definition of the derivative says that 
\[
\frac{d}{dx} f(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.
\]
In this case, we get
\[
\lim_{h \to 0} \frac{(x+h)^2 - 5(x+h) - (x^2 - 5x)}{h} = \lim_{h \to 0} \frac{x^2 + 2xh + h^2 - 5x - 5h - x^2 + 5x}{h} = \lim_{h \to 0} \frac{2xh + h^2 - 5h}{h} = \lim_{h \to 0} 2x + h - 5 = 2x - 5,
\]
which is the answer you would expect from the rules of differentiation.

6. (a) State the \( \delta-N \) definition of \( \lim_{x \to a} f(x) = \infty \). (5 points)

**Solution:** For any \( N > 0 \), there exists \( \delta > 0 \) such that \( |x - a| < \delta \) implies \( f(x) > N \).

(b) Use the \( \delta-N \) definition of a limit to prove that \( \lim_{x \to 3} \frac{1}{(x-3)^2} = \infty \). (15 points)

**Solution:** To find an appropriate choice of \( \delta \), you would do the usual working backwards process from the inequality involving \( N \) to the inequality involving \( x-3 \). Then your proof could read as follows.

**Proof.** Given \( N > 0 \), let \( \delta = \frac{1}{\sqrt{N}} \). We must prove that \( |x - 3| < \delta \) implies \( \frac{1}{(x-3)^2} > N \). So assume that \( |x - 3| < \delta = \frac{1}{\sqrt{N}} \). Then
\[
-\frac{1}{\sqrt{N}} < x-3 < \frac{1}{\sqrt{N}}, \text{ so} \quad 0 < (x-3)^2 < \frac{1}{N}, \text{ so} \quad \frac{1}{(x-3)^2} > N,
\]
which is what we wanted to prove. \( \square \)

7. Show that the function \( f(x) = |x-3| \) is not differentiable at \( x = 3 \). (15 points)

**Solution:** The graph of this function looks like \( y = |x| \), but shifted 3 units to the right. The graph has a sharp corner at \( x = 3 \), which is an indication that it is not differentiable. To show the function is not differentiable, you either need to make a detailed argument about slopes of secant lines, or you need to explain why one of the limits that can be used to define the derivative does not exist.

**Geometric argument:** Refer to the graph of \( f \) shown below. The derivative of \( f \) at \( x = 3 \) would be the limit of the slopes of secant lines through \((3,0)\). A secant line through \((3,0)\) and a point on the graph of \( f \) with \( x > 3 \) will have positive slope. In fact, it will coincide with the portion of the graph of \( f \) where \( x \geq 3 \). Therefore, it will have slope 1. A secant line through \((3,0)\) and a point on the graph of \( f \) with \( x < 3 \) will have negative slope. It will coincide with the portion of the graph of \( f \) where \( x \leq 3 \), so it will have slope \(-1\). So, if we consider the limits of slopes of secant lines through
(3, 0) and \((x, f(x))\) as \(x \to 3\), we will obtain 1 letting \(x\) approach 3 from the right, but -1 letting \(x\) approach 3 from the left. In other words, the limit of slopes of such secant lines does not exist. Therefore, the derivative of \(f\) at \(x = 3\) does not exist.

*Algebraic argument:* The derivative of \(f\) at \(x = 3\) is defined to be the limit

\[
\lim_{h \to 0} \frac{f(x) - f(a)}{x - a} = \lim_{x \to 3} \frac{|x - 3| - |3 - 3|}{x - 3} = \lim_{x \to 3} \frac{|x - 3|}{x - 3}.
\]

We will show that this limit does not exist, hence that the derivative of \(f\) at \(x = 3\) does not exist, by showing that the left and right limits do not match. If \(x > 3\), then \(x - 3 > 0\), so

\[
\lim_{x \to 3^+} \frac{|x - 3|}{x - 3} = \lim_{x \to 3^+} \frac{x - 3}{x - 3} = 1.
\]

If \(x < 3\), then \(x - 3 < 0\), so

\[
\lim_{x \to 3^-} \frac{|x - 3|}{x - 3} = \lim_{x \to 3^-} \frac{-(x - 3)}{x - 3} = -1.
\]
8. Draw a graph of a function $f$ with the following properties. Label your graph carefully to identify the properties. (3 points each property = 15 points)

- $f$ does not have an inverse
- $\lim_{x \to \infty} f(x) = -3$
- $f$ has a vertical asymptote at $x = -1$
- the slope of the tangent line to $f$ at 0 is zero
- $f$ has a removable discontinuity

Solution: There are lots of graphs that you could draw as a solution to this question. Whatever you draw, it should fail the horizontal line test somewhere, have a horizontal asymptote (as $x \to \infty$) at $y = -3$, have a vertical asymptote at $x = -1$, have a horizontal tangent line at $x = 0$, and have a hole somewhere. Below is an example.
EXTRA CREDIT

EC1) Suppose that
\[ f(x) = \begin{cases} 
-1 & \text{if } x < -3 \\
 x + 2 & \text{if } -2 \leq x \leq 1 \\
3 & \text{if } x > 1 
\end{cases} \]
and calculate \( \lim_{x \to 0} f(x)(xe^x - x^3) \).

**Solution:** There are (at least) two ways to solve this problem. First, you could recognize that limits only depend on the value of the function near the point where the limit is being evaluated. Since we’re evaluating a limit as \( x \to 0 \), we can replace \( f(x) \) with \( x + 2 \), since they’re the same near 0. (You had to explicitly argue this in your solution to receive full credit.) Then we evaluate as follows, using the fact that polynomials and exponential functions are continuous.

\[
\lim_{x \to 0} f(x)(xe^x - x^3) = \lim_{x \to 0} (x + 2)(xe^x - x^3) = \left( \lim_{x \to 0} x + 2 \right) \left( \lim_{x \to 0} xe^x - x^3 \right) = (0 + 2)(0 \cdot e^0 - 0^3) = 0
\]

Alternatively, you could use the squeeze theorem. By looking at the equation for \( f \) or by graphing \( f \), you can see that \( f \) is bounded between \(-1\) and \(3\). Therefore,

\[-(xe^x - x^3) \leq f(x)(xe^x - x^3) \leq 3(xe^x - x^3)\]

Evaluate the limits of the functions on the left and right by direct substitution (since everything in sight is continuous):

\[
\lim_{x \to 0} -(xe^x - x^3) = 0 \quad \text{and} \quad \lim_{x \to 0} 3(xe^x - x^3) = 0.
\]

Therefore, by the squeeze theorem,

\[
\lim_{x \to 0} f(x)(xe^x - x^3) = 0.
\]

EC2) Use the \( \epsilon-\delta \) definition of a limit to prove that
\[
\lim_{x \to 3} x^2 = 9.
\]

**Solution 1:** The implication we’re working with in this problem is

\[
0 < |x - 3| < \delta \quad \text{implies} \quad |x^2 - 9| < \epsilon.
\]

For small values of \( \epsilon \), the following method will work to find an appropriate \( \delta \). Start with the \( \epsilon \) side of the implication and work backwards to get an inequality involving \( x - 3 \):

\[
\begin{align*}
|x^2 - 9| &< \epsilon \\
-\epsilon &< x^2 - 9 < \epsilon \\
9 - \epsilon &< x^2 < \epsilon + 9 \\
\sqrt{9 - \epsilon} &< x < \sqrt{\epsilon + 9} \\
\sqrt{9 - \epsilon} - 3 &< x - 3 < \sqrt{\epsilon + 9} - 3
\end{align*}
\]
For small values of $\epsilon$, the number on the far left above is negative and the number on the far right is positive. However, we don’t know which of these numbers is smaller in absolute value. To get around this, just choose $\delta = \min(3 - \sqrt{9 - \epsilon}, \sqrt{\epsilon + 9} - 3)$. Now for the proof.

**Proof.** Given $\epsilon > 0$, choose

$$\delta = \min(3 - \sqrt{9 - \epsilon}, \sqrt{\epsilon + 9} - 3).$$

Let’s assume that $|x - 3| < \delta$. Then

\[
\begin{align*}
    x - 3 &< \sqrt{\epsilon + 9} - 3, \\
    x &< \sqrt{\epsilon + 9}, \\
    x^2 &< \epsilon + 9, \\
    x^2 - 9 &< \epsilon,
\end{align*}
\]

which is one part of what we needed to prove. For the other part, $|x - 3| < \delta$ also implies $-\delta < x - 3$, so

\[
\begin{align*}
    \sqrt{9 - \epsilon} - 3 &< x - 3, \\
    \sqrt{9 - \epsilon} &< x, \\
    9 - \epsilon &< x^2, \\
    -\epsilon &< x^2 - 9.
\end{align*}
\]

These two calculations together tell us that $|x^2 - 9| < \epsilon$, which is what we needed to prove.

**Solution 2:** A second way to prove this limit is to factor $|x^2 - 9|$ as $|x + 3||x - 3|$, then bound $|x + 3|$. For example, you might write the following proof.

**Proof.** Given $\epsilon > 0$, let $\delta = \min(\frac{\epsilon}{7}, 1)$. Now assume that $|x - 3| < \delta$ and show that $|x^2 - 9| < \epsilon$. Since $|x - 3| < \delta \leq 1$, we have $2 < x < 4$, which means $5 < x + 3 < 7$, so certainly $|x + 3| < 7$. We also know that $|x - 3| < \delta \leq \frac{\epsilon}{7}$. Therefore,

\[
|x^2 - 9| = |x + 3||x - 3| < 7 \cdot \frac{\epsilon}{7} = \epsilon.
\]

\[\square\]

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1 A technical aside: This choice of $\delta$ will work so long as $\epsilon < 9$. Notice that if $\epsilon > 9$, then one of the quantities in this definition of $\delta$ involves taking the square root of a negative number. However, if $\epsilon > 9$, then it is easy to find a $\delta$ that makes the implication true. Just choose $\delta = 1$. If $|x - 3| < 1$, then $x$ is between 2 and 4, so $|x^2 - 9|$ is between 5 and 7, so certainly $|x^2 - 9| < \epsilon$. Really, then, we should choose $\delta = \min(3 - \sqrt{9 - \epsilon}, \sqrt{\epsilon + 9} - 3, 1)$.

2 And here we’re assuming also that $\epsilon < 9$. To be extra careful about the proof, you could split it into two cases. The $\epsilon < 9$ case would read as in the main text here. The $\epsilon > 9$ case would just be the justification given in the previous footnote. This problem with larger values of $\epsilon$ is the same reason that your textbook chooses $\delta = \min(1, \text{something else})$ in this situation, even though they solve these sorts of problems using a different method overall.