Calculus I
Practice Midterm I

Name:______________________________

Instructions: You have 75 minutes to complete the exam. Calculators, notes and textbooks are not allowed. Provide the answers in the simplest possible form that does not require calculator use (e.g. expressions like $\sqrt{7}$ are fine). Show all of your work. If you only give a numerical answer you will receive no credit. Partial credit will be given for partial solutions.
Write your solutions in the space below the questions. If you need more space, use the back of the page. Do not forget to write your name in the space above.

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1. Find the domain and range of the following functions. Sketch the graph of (b).

(a) $f(x) = \sin \left( \frac{x}{(x-2)(x-3)} \right)$
(b) $f(x) = 2\arccos(x - 2) + 1$
(c) $f(x) = \ln(x^2 - 3x + 2)$

**Solution:** (a) The function $f(x)$ is the composition of the functions $g(x) = \sin(x)$ and $h(x) = \frac{x}{(x-2)(x-3)}$. As we saw in class, the function $g(x) = \sin(x)$ is defined for every real number $x$ and its range is $[-1, 1]$. The domain of $h(x) = \frac{x}{(x-2)(x-3)}$ is $\mathbb{R} \setminus \{2, 3\}$. Therefore the domain of $f(x)$ equals $\mathbb{R} \setminus \{2, 3\}$ and its range is the closed interval $[-1, 1]$.

(b) The function $g(x) = \arccos(x)$ has domain the closed interval $[-1, 1]$ and range the closed interval $[0, \pi]$. Hence the domain of $f(x)$ is the closed interval $[1, 3]$ and its range is the closed interval $[1, 2\pi + 1]$.

The graph of $f(x)$ looks like the one for $g(x)$ (which we saw in class), translated to the right by 2 units, stretched vertically by a factor of 2 and translated up by 1 unit (but you should draw it, not just describe it!).

(c) The domain of the function $g(x) = \ln(x)$ is the open interval $(0, \infty)$. Thus, to find the domain of $f(x)$, we have to find all the $x$ such that $x^2 - 3x + 2 > 0$. Now $x^2 - 3x + 2 = (x-1)(x-2)$, so the graph of the function $h(x) = x^2 - 3x + 2$ crosses the $x$-axis at $x = 1$ and $x = 2$. It follows that $x^2 - 3x + 2 > 0$ exactly when $x < 1$ or $x > 2$. Hence the domain of $f(x)$ is the set of all real numbers $x$ such that $x < 1$ or $x > 2$. 


2. (a) State the $\epsilon - \delta$ definition of limit.
(b) Use it to prove $\lim_{x \to 2} 4x + 1 = 9$.

Solution: (a) We say that $\lim_{x \to a} f(x) = L$ if for every $\epsilon > 0$, there exists $\delta > 0$ such that if $0 < |x - a| < \delta$, then $|f(x) - L| < \epsilon$.

(b) Consider some $\epsilon > 0$. Then:

$$|4x + 1 - 9| < \epsilon \Rightarrow |4x - 8| < \epsilon \Rightarrow |x - 2| < \frac{\epsilon}{4}$$

Thus, if we take $\delta = \frac{\epsilon}{4}$, we get that

$$0 < |x - 2| < \delta \Rightarrow 4|x - 2| < \epsilon \Rightarrow |4x - 8| < \epsilon \Rightarrow |4x + 1 - 9| < \epsilon$$

and this shows that $\lim_{x \to 2} 4x + 1 = 9$. 
3. Calculate the following limits:
   (a) \( \lim_{x \to 0^+} \frac{\sqrt{x^2+1} - 1}{x} \)
   (b) \( \lim_{x \to 0} x^2 \cos(\frac{1}{x}) \)
   (c) \( \lim_{x \to -2} \frac{x+3}{x^2+5x+6} \)

**Solution:**
(a) We rationalize the expression inside the limit:
\[
\frac{\sqrt{x^2+1} - 1}{x} = \frac{\sqrt{x^2+1} - 1}{x} \cdot \frac{\sqrt{x^2+1} + 1}{\sqrt{x^2+1} + 1} = \frac{x^2+1 - 1}{x(\sqrt{x^2+1} + 1)} = \frac{x}{\sqrt{x^2+1} + 1}
\]
and then we have:
\[
\lim_{x \to 0^+} \frac{\sqrt{x^2+1} - 1}{x} = \lim_{x \to 0^+} \frac{x}{\sqrt{x^2+1} + 1} = \lim_{x \to 0^+} \frac{x}{\sqrt{x^2+1} + 1} = \frac{0}{2} = 0
\]

(b) First note that for every \( x \neq 0 \), we have
\[-1 \leq \cos\left(\frac{1}{x}\right) \leq 1\]
Multiplying this by \( x^2 \) we get:
\[-x^2 \leq x^2 \cos\left(\frac{1}{x}\right) \leq x^2\]
Now \( \lim_{x \to 0^-} -x^2 = \lim_{x \to 0^+} x^2 = 0 \), so by the Squeeze Theorem we conclude that \( \lim_{x \to 0} x^2 \cos\left(\frac{1}{x}\right) = 0 \).

(c) By the quadratic formula, we find that the solutions to \( x^2 + 5x + 6 \) are \( x = -2 \), \( x = -3 \). Hence we can write
\[
\frac{x+3}{x^2+5x+6} = \frac{x+3}{(x+2)(x+3)} = \frac{1}{x+2}
\]
We have
\[
\lim_{x \to -2^+} \frac{x+3}{x^2+5x+6} = \lim_{x \to -2^+} \frac{1}{x+2} = +\infty
\]
\[
\lim_{x \to -2^-} \frac{x+3}{x^2+5x+6} = \lim_{x \to -2^-} \frac{1}{x+2} = -\infty
\]
and hence the limit does not exist.
4. (a) Calculate the following limit: \( \lim_{x \to 3} \frac{2x^3 - 6x^2}{x - 3} \).
(b) Using that \( \lim_{x \to 0} \frac{e^x - 1}{x} = 1 \), calculate \( \lim_{x \to 0} \frac{e^{2x} - 1}{x} \).

Solution: (a) We first simplify the expression inside the limit:
\[
\frac{2x^3 - 6x^2}{x - 3} = \frac{2x^2(x - 3)}{x - 3} = 2x^2
\]
Then we have:
\[
\lim_{x \to 3^+} 2x^2 = 18
\]
\[
\lim_{x \to 3^-} 2x^2 = -18
\]
Since the two side limits are different, the limit does not exist.

(b) We have:
\[
\lim_{x \to 0} \frac{e^{2x} - 1}{x} = \lim_{x \to 0} \frac{(e^x - 1)(e^x + 1)}{x} = \lim_{x \to 0} \frac{e^x - 1}{x} \cdot \lim_{x \to 0} (e^x + 1) = 1 \cdot 2 = 2.
\]
5. Given is the graph of $f(x)$:

![Graph of $f(x)$](image)

State the value of the following limits:

- $\lim_{x \to -2^{-}} f(x)$
- $\lim_{x \to -2^{+}} f(x)$
- $\lim_{x \to -1^{-}} f(x)$
- $\lim_{x \to -1^{+}} f(x)$

**Solution:**

- $\lim_{x \to -2^{-}} f(x) = +\infty$
- $\lim_{x \to -2^{+}} f(x) = 1$
- $\lim_{x \to -1^{-}} f(x) = 2$
- $\lim_{x \to -1^{+}} f(x) = 2$
6. Where are the following functions continuous? Justify your answer by calculating limits, drawing graphs or referring to theorems as appropriate.

(a) \( f(x) = \cos(x^2) + x^3 \)  
(b) \( f(x) = |2x - 5| \)  
(c) \( f(x) = \frac{x^3 + e^{\sin(x)}}{(x - 1)\cos(x)} \)

**Solution:**

(a) The function \( f_1(x) = x^3 \) is a polynomial and hence it is continuous at every real number \( x \). The function \( \cos(x^2) \) is a composition of the functions \( f_2(x) = \cos(x) \) and \( f_3(x) = x^2 \), which (as we saw in class) are continuous everywhere. By the theorem on the continuity of composite functions, the function \( \cos(x^2) \) is continuous at every real number \( x \).

Thus \( f(x) \) is a sum of two functions that are continuous everywhere and hence (by the theorem on continuity for sums of functions), \( f(x) \) is continuous at every real number \( x \).

(b) The function \( f(x) \) is a composition of the function \( f_1(x) = |x| \) and \( f_2(x) = 2x - 5 \). The function \( f_2(x) \), being a polynomial, is continuous at every real number \( x \). From the graph of \( f_1(x) \) (draw it!) we see that it is also continuous at every number \( x \). Therefore, by the theorem on continuity for composite functions, we conclude that \( f(x) \) is continuous at every \( x \).

(c) The function \( f(x) \) is a quotient, hence it is continuous exactly at the points where both the numerator and denominator are continuous and the denominator is non-zero. Now:

\[
(x - 1)\cos(x) = 0 \iff x = 1 \text{ or } x = 2n\pi, \quad n \in \mathbb{Z}
\]

The numerator is a sum of a polynomial term \( x^3 \) (continuous everywhere) with a term \( e^{\sin(x)} \) that is a composition of functions continuous everywhere, hence also everywhere continuous. Therefore the numerator is continuous everywhere. Similarly the denominator is a product of functions continuous everywhere, hence it is also continuous everywhere. It follows that the function \( f(x) \) is continuous at every point of its domain, which is \( \mathbb{R} \) except the points \( x = 1 \) and \( x = 2n\pi \) for any \( n \in \mathbb{Z} \).
7. (a) State what it means for a function $f(x)$ to be continuous at $a$.
(b) Give an example of a function $f(x)$ that is continuous at every point of its domain. (No need for anything fancy here!).

**Solution:** (a) We say that $f(x)$ is continuous at $a$ if $\lim_{x \to a} f(x) = f(a)$.

(b) An example is $f(x) = 0$. Other examples are any polynomial or trigonometric function.
8. Consider the function

\[ f(x) = \begin{cases} 
  x + 2, & x \geq 1 \\
  -x^2 + a, & -1 < x < 1 \\
  \frac{1}{4}a|x| + b, & x \leq -1 
\end{cases} \]

(a) For what values of \(a\) is it continuous at \(x = 1\)?
(b) Are there some values of \(a\) and \(b\) making it continuous both at \(x = -1\) and \(x = 1\)?

**Solution:** (a) For \(f(x)\) to be continuous at \(x = 1\) we need \(\lim_{x \to 1} f(x) = f(1) = 1 + 2 = 3\). We have:

\[ \lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} x + 2 = 3 \]

\[ \lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} -x^2 + a = -1 + a \]

Therefore, for \(f(x)\) to be continuous at \(x = 1\), we need \(-1 + a = 3\), that is, \(a = 4\).

(b) By part (a), we need to take \(a = 4\) to make \(f(x)\) continuous at \(x = 1\). With \(a = 4\), we have:

\[ \lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} -x^2 + 4 = 3 \]

\[ \lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} \frac{1}{4}|x| + b = 1 + b \]

hence \(f(x)\) will be continuous at \(x = -1\) if we take \(b\) such that \(1 + b = 3\), that is, \(b = 2\).