Problem 1. Let

\[ H(x) = \frac{1}{\sqrt{x^2 - 5x}}. \]

(a) (3 points) Find functions \( f, g, \) and \( h \) such that \( H(x) = f \circ g \circ h(x) \).

*Solution.* There is more than one correct answer. One answer is

\[ f(x) = \frac{1}{x}, \quad g = \sqrt{x}, \quad h(x) = x^2 - 5x. \]

(b) (3 points) What is the domain of \( H(x) \)? Explain why.

*Solution.* The domain is when \( x^2 - 5x > 0 \); that is, \( x(x - 5) > 0 \). We have two cases: either

\[ x > 0 \text{ and } x - 5 > 0 \]

or

\[ x < 0 \text{ and } x - 5 < 0. \]

Therefore, the domain is

\[ \{ x : x > 5 \text{ or } x < 0 \}. \]

(c) (1 bonus point) What is the range of \( H(x) \)? Explain why.

*Solution.* We show that the range is \((0, \infty)\). The reason is the following: it is easy to see that the range of \( h \) contains \([0, \infty)\) by drawing the graph of \( h \) (or by the Intermediate Value Theorem). Then the range of \( g \circ h \) is also \([0, \infty)\). After we compose \( g \circ h \) with the reciprocal function \( f \), we can then conclude the range of \( H = f \circ g \circ h \) is \((0, \infty)\). It does not contain 0 because there is no real number \( x \) so that \( H(x) = 0 \).
Problem 2. (12 points, 4 points each) Find the limit, if it exists. If the limit does not exist, explain why.

(a) $\lim_{x \to 1} (x - 1)^2 \sin \left( \frac{2}{x - 1} \right)$.

Solution.  
\[-(x - 1)^2 \leq (x - 1)^2 \sin \left( \frac{2}{x - 1} \right) \leq (x - 1)^2.\]

Because \[\lim_{x \to 1} -(x - 1)^2 = 0 = \lim_{x \to 1} (x - 1)^2,\]
by the Squeeze Theorem, we have \[\lim_{x \to 1} (x - 1)^2 \sin \left( \frac{2}{x - 1} \right) = 0.\]

(b) $\lim_{x \to 0} \left( \frac{1}{x \sqrt{1 + x}} - \frac{1}{x} \right)$.

Solution.  
\[
\frac{1}{x \sqrt{1 + x}} - \frac{1}{x} = \frac{1 - \sqrt{1 + x}}{x \sqrt{1 + x}} = \frac{1 - \sqrt{1 + x}}{x(x \sqrt{1 + x})(1 - \sqrt{1 + x})} = \frac{-1}{(1 + \sqrt{1 + x})(1 + \sqrt{1 + x})}.
\]
So, \[\lim_{x \to 0} \left( \frac{1}{x \sqrt{1 + x}} - \frac{1}{x} \right) = \lim_{x \to 0} \frac{-1}{(1 + \sqrt{1 + x})(1 + \sqrt{1 + x})} = -\frac{1}{2}.
\]

(c) $\lim_{x \to \infty} \frac{1}{x} \sin \left( \frac{1}{x} \right)$.

Solution. Because \[\lim_{x \to \infty} \frac{1}{x} = 0, \quad \text{and} \quad \lim_{x \to \infty} \sin \left( \frac{1}{x} \right) = \sin \left( \lim_{x \to \infty} \frac{1}{x} \right) = \sin(0) = 0,\]
by product law, \[\lim_{x \to \infty} \frac{1}{x} \sin \left( \frac{1}{x} \right) = 0.
\]
Problem 3. Let \( f(x) = \ln x \) for \( x > 0 \).

(a) (4 points) Let \( g(x) = f(|x|) \). Draw the graph of \( y = g(x) \), and find the domain of \( g(x) \).

Solution. \( g = \ln |x| \) and its domain is \((-\infty, 0) \cup (0, \infty)\). The graph \( y = \ln |x| \) is the following:

(b) (6 points) Starting with the graph of \( y = g(x) \) from part (a). Find the equation of the graph of that results from reflecting about \( x = 2 \).

Solution. From the graph below, we see that the new graph is obtained from shifting the old graph 4 units to the right, so the equation of the new graph is \( y = \ln |x - 4| \).
Problem 4. Let
\[ f(x) = \begin{cases} 
\frac{1}{4}x^2 + \frac{3}{4} & \text{if } x < 1 \\
\sqrt{x} & \text{if } x \geq 1 
\end{cases} \]

(a) (2 points) Show that \( f \) is continuous on \((-\infty, \infty)\).

Solution. When \( x < 1 \), \( f(x) \) is a polynomial, it is continuous. When \( x > 1 \), \( f(x) \) is a root function, so it is continuous. Then we only need to check that at the point \( x = 1 \), the one-sided limits are equal:
\[ \lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} \sqrt{x} = 1 \]
and
\[ \lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} \left( \frac{1}{4}x^2 + \frac{3}{4} \right) = 1. \]
Therefore, \( f \) is continuous on \((-\infty, \infty)\).

(b) (5 points) Is \( f \) differentiable on \((-\infty, \infty)\)? If yes, prove it. If not, find the points where \( f \) is not differentiable.

Solution. When \( x < 1 \),
\[ f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{(x+h)^2 + 3 - (x^2 + 3)}{4h} = \frac{1}{2}x, \]
so \( f'(x) \) exists for any \( x < 1 \). When \( x > 1 \),
\[ f'(x) = \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} = \lim_{h \to 0} \frac{x+h-x}{h(\sqrt{x+h} + \sqrt{x})} = \frac{1}{2\sqrt{x}}. \]
Then \( f'(x) \) exists for any \( x > 1 \). It remains to check that \( f'(1) \) exists. Because the one-sided limits are equal:
\[ \lim_{h \to 0^-} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0^-} \frac{(1+h)^2 + 3 - (1^2 + 3)}{4h} = \frac{1}{2}, \]
and
\[ \lim_{h \to 0^-} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0^-} \frac{\sqrt{1+h} - \sqrt{1}}{h} = \frac{1}{2}, \]
\( f'(1) \) exists. Therefore, \( f \) is differentiable on \((-\infty, \infty)\).
(c) (continued from previous page, 5 points) Find the equation of the tangent line to the curve $y = f(x)$ at $x = 0$.

Solution. Assume that the equation of the tangent line is $y = mx + b$ where $m$ is the slope and $b$ is the $y$–intercept. Then

$$m = f'(0) = \frac{1}{2} \cdot 0 = 0$$

from part (b), so the tangent line is horizontal. Also, $f(0) = 3/4$ from the definition of $f$, so the tangent line passes through $(0, 3/4)$. Then, the equation of the tangent line is

$$y = \frac{3}{4}.$$
Problem 5. Let
\[ f(x) = \frac{1 - 2e^x}{1 + 2e^x}. \]

(a) (6 points) \( f \) is a one-to-one function (you do not need to check it). Find the inverse function \( f^{-1} \) of \( f \), and state the domain of \( f^{-1} \).

Solution. Let
\[ y = \frac{1 - 2e^x}{1 + 2e^x} \Rightarrow (1 + 2e^x)y = 1 - 2e^x \Rightarrow e^x = \frac{1 - y}{2y + 2}. \]
Taking natural logarithms of both sides of this equation, we get
\[ x = \ln \left( \frac{1 - y}{2y + 2} \right). \]
Now, replace \( y \) by \( x \), we get
\[ f^{-1}(x) = \ln \left( \frac{1 - x}{2x + 2} \right). \]
The domain of \( f^{-1} \) is those \( x \)'s which satisfy
\[ \frac{1 - x}{2x + 2} > 0; \]
that is either
\[ 1 - x > 0 \text{ and } 2x + 2 > 0 \]
or
\[ 1 - x < 0 \text{ and } 2x + 2 < 0. \]
We then get the domain is \((-1, 1)\).

(b) (4 points) Show that there is a negative solution of the equation \( f(x) = 0 \).

Solution.

Method 1: We have found the inverse function from part (a), because if \( a \) is a solution so that \( f(a) = 0 \) if and only if \( f^{-1}(0) = a \). Then
\[ f^{-1}(0) = \ln \left( \frac{1 - 0}{0 + 2} \right) = \ln \frac{1}{2} < 0. \]

Method 2: We can apply the Intermediate Value Theorem because \( f \) is a continuous function.
\[ f(0) = \frac{1 - 2}{1 + 2} < 0. \]
When \( x \) is very negative, say \( x = -10 \), we have
\[ f(-1) = \frac{1 - 2e^{-10}}{1 + 2e^{-10}} > 0 \]
because \( 2e^{-10} < 1 \). Another way to see this is, \( \lim_{x \to -\infty} e^x = 0 \), so for some negative \( x \), we must have \( 2e^x < 1 \).
Then the Intermediate Value Theorem says there is a solution to $f(x) = 0$ on $(0, -10)$. 