The tangent line to the curve \( y = f(x) \) at the point \( P(a, f(a)) \)
is the line through \( P \) with slope

\[
m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}
\]

provided this limit exists.

Example:

Find the equation of the tangent line to the parabola \( y = x^2 \)
at the point \( P(1,1) \).

\[
m = \lim_{x \to 1} \frac{x^2 - 1}{x - 1} = 2
\]

\[
y - 1 = 2(x - 1) \quad \Rightarrow \quad y = 2x - 2 + 1 \quad \Rightarrow \quad y = 2x - 1
\]
\[ m = \lim_{h \to 0} \frac{f(e+h) - f(e)}{h} \]

\[ m_{PA} = \frac{f(e+h) - f(e)}{h} \]

\[ \text{as } x \to e \quad x = e + h \]

\[ h \to 0 \quad h = e - a \]

**Example**

Find the equation of the tangent line to the hyperbola \( y = \frac{2}{x} \) at the point \((3, 1)\).

\[ m = \lim_{h \to 0} \frac{f(3+h) - f(3)}{h} \]

\[ m = \lim_{h \to 0} \frac{\frac{2}{3+h} - \frac{2}{3}}{h} = -\frac{1}{3} \]

\[ y - 1 = -\frac{1}{3} (x - 3) = -\frac{x}{3} + 1 \]

\[ y = -\frac{1}{3}x + 2 \]
Velocities:

\[ s = f(t) - f(0) \]

\[ \text{Average velocity} = \frac{\text{displacement}}{\text{time}} = \frac{f(t) - f(0)}{h} \]

\[ v(t) \quad \text{Instantaneous velocity} = \lim_{h \to 0} \frac{f(t + h) - f(t)}{h} \]

This means that the velocity at time \( t = 2 \) is equal to the slope of the tangent line at \( P(2, f(2)) \).

A particle moves along a straight line with equation of motion \( s(t) = t^2 - 6t + 5 \), where \( s(t) \) is measured in meters and \( t \) in seconds. Find velocity when \( t = 2 \).

\[ v(t) = \lim_{h \to 0} \frac{s(2 + h) - s(2)}{h} = \lim_{h \to 0} \frac{(2 + h)^2 - 6(2 + h) + 5 - (4 - 12 + 5)}{h} \]

\[ = \lim_{h \to 0} \frac{h^2 + 4h + 4 - 12 - 6h + 4 + 12}{h} = \lim_{h \to 0} \frac{h^2 - 8h}{h} \]

\[ = \lim_{h \to 0} \frac{h(h - 8)}{h} = \lim_{h \to 0} (h - 8) = -2 \text{ m/sec} \]
In general \( y = f(x) \)

\[
\Delta x = x_2 - x_1 \\
\Delta y = f(x_2) - f(x_1) \\
\Delta y = \frac{f(x_2) - f(x_1)}{x_2 - x_1}
\]

\( \Delta x \to 0 \) as \( x_2 \to x_1 \)

**Average rate of change of \( f \) wrt \( x \) over the interval \([x_1, x_2]\)**

**Instantaneous rate of change**

Since this limit seems to occur in many phenomena of science or engineering, it is given a special name and notation—

**Definition:** The derivative of a function \( f \) at \( a \) is the limit, denoted by \( f'(a) \), given by

\[
f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}
\]

if this limit exists.

- If \( f'(a) \) exists, we say that \( f \) is differentiable at \( a \).
- \( f \) is differentiable on \((b, c)\) if it is diff. at every point in the interval.

**Problem 24** p. 163 (PS 4)
Example 1:

The \( \lim_{h \to 0} \frac{\sqrt{1+h} - 1}{h} \) is equal to \( f'(x) \) for some function \( f(x) \) and some value of \( a \). What is \( f(x) \) and what is \( a \)?

\[
\begin{align*}
\frac{f(h+a) - f(a)}{h} &= \frac{\sqrt{1+h} - 1}{h} \\
f(a+h) &= \sqrt{1+h} \\
f(a) &= 1 \\
\text{Take } f(x) &= \sqrt{x} - a = 1
\end{align*}
\]

Example 2:

Find the derivative of \( f(x) = x^3 - x^2 + 2x \) using the def. of derivative.

At which point? For a generic \( x \), sub \( a \) by \( x \).

\[
f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}
\]

\[
= \lim_{h \to 0} \frac{(x+h)^3 - (x+h)^2 + 2(x+h) - x^3 + x^2 - 2x}{h}
\]

\[
= \lim_{h \to 0} \frac{x^3 + 3x^2h + 3xh^2 + h^3 - x^3 - 2x - x^2 + 2x - x^3 + x^2 - 2x}{h}
\]

\[
= \lim_{h \to 0} \frac{3x^2h + 3xh^2 - 2xh + h^3 - h^2}{h}
\]

\[
= \lim_{h \to 0} \frac{h(3x^2 + 3x - 2x + h^2 - h)}{h}
\]

\[
= \lim_{h \to 0} \frac{3x^2 - 2x + 3xh + h^2 - h}{h}
\]

\[
= 3x^2 - 2x
\]
Given a function \( f \), we have a new function \( f' \), called the derivative, defined as:

\[
 f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}
\]

**Remark 1:**

If \( f \) is differentiable at \( a \), then \( f \) is also continuous at \( a \).

**Proof:**

We want to show \( \lim_{x \to a} f(x) = f(a) \) or \( \lim_{x \to a} f(x) - f(a) = 0 \).

\[
\lim_{x \to a} (f(x) - f(a)) = \lim_{x \to a} \left( f(x) - f(a) \right) 
\]

\[
= \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \cdot (x - a) = 0
\]

\[
\downarrow
\]

\[
\frac{f'(a)}{0}
\]

**Example:**

If the function \( f(x) \) is defined as:

\[
f(x) = \begin{cases} 
 1 & \text{if } x > 2 \\
 0 & \text{if } x \leq 2 
\end{cases}
\]

**Graph:**

\[
\begin{array}{c}
\text{X > 2} \\
\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{1 - 1}{h} = 0
\end{array}
\]

\[
\begin{array}{c}
\text{X < 2} \\
\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{-1 - (-1)}{h} = 0
\end{array}
\]

\[
\begin{array}{c}
\text{X = 2} \\
\lim_{h \to 0} \frac{1 - 1}{h} = 0
\end{array}
\]

**Draw graph of \( f' \).**

**No**

**Why?**
Here is a graph of $f$.

Draw graph of $f'$.