For \( f \geq 0 \), \( g(x) = \text{AREA below } f \text{ so far} \)

\[
g(x) = \int_a^x f(t) \, dt
\]

\[
g(x+h) = \int_a^{x+h} f(t) \, dt
\]

\[
g(x) \approx f(x) \cdot h
\]

\[
g'(x) = \lim_{h \to 0} \frac{g(x+h) - g(x)}{h} \approx f(x)
\]

**FTC (Part 1):** If \( f \) is \( \text{cts} \) on \([a, b] \), then the function \( g \) defined by

\[
g(x) = \int_a^x f(t) \, dt \quad a \leq x \leq b
\]

is \( \text{cts} \) on \([a, b] \) and \( \text{diff} \) on \((a, b)\), and \( g'(x) = f(x) \).
Supp. \( F \) is an antiderivative of \( f \) : \( F' = f \cdot g' \).

Then \( F(x_1) = g(x_1) + C \) for \( a \leq x \leq b \).

But both \( F, g \) are ds. on \([a,b]\) and so,

\[ F(x) = g(x) + C \quad \text{for all } x \in [a,b]. \]

Note \( g(e) = \int_{e}^{1} f(t) \, dt = 0 \).

\[
\begin{align*}
F(b) - F(a) &= g(b) + C - (g(e) + C) \\
           &= g(b) + C - C \\
           &= g(b) = \int_{e}^{1} f(t) \, dt.
\end{align*}
\]
Thm: If f is either continuous or monotone on \([a, b]\) \(\square\)
then \(f\) is integrable on \([a, b]\); that is \(\int_a^b f(x) \, dx\) exists.

How do we compute the definite integral of \(f\)?
We can either use the definition (and do the sums --)
and then the limit or use the following.

The Fundamental Theorem of Calculus: \(\square\) Repeat

Suppose \(f\) is continuous on \([a, b]\).

(1) For any \(a \leq x \leq b\)
\[ g(x) = \int_a^x f(t) \, dt \] Then
\(g\) is continuous on \([a, b]\), and \(g'(x) = f(x)\) —

(2) \(\int_a^b f(x) \, dx = F(b) - F(a)\)
where \(F\) is any antiderivative of \(f\); that is \(F' = f\).

Ex) Calculate \(\int_0^1 x^2 \, dx\).
\[= \frac{1}{3} x^3 \bigg|_0^1 = \frac{1}{3} - \frac{1}{3} = \frac{1}{3} \]
Example: Evaluate \( \int_{-2}^{1} x^3 \, dx \)

\[
\int_{-2}^{1} x^3 \, dx = F(1) - F(-2) = \left[ \frac{1}{4} (1)^4 + C \right] - \left[ \frac{1}{4} (-2)^4 + C \right] = \frac{1}{4} (1)^4 - \frac{1}{4} (-2)^4 = -\frac{15}{4}.
\]

\( F(x) = \frac{1}{4} x^4 + C \)

Example 2:

\[
\int_{-1}^{3} (3x^2 - x + 6) \, dx \quad \text{cts. fct'n on } [-1, 3]
\]

\[
\int_{-1}^{3} (3x^2 - x + 6) \, dx = \left[ x^3 - \frac{x^2}{2} + 6x \right]_{-1}^{3} = (3)^3 - \frac{(3)^2}{2} + 6(3) - \left( (-1)^3 - \frac{(-1)^2}{2} + 6(-1) \right) = \frac{27}{2} + \frac{9}{2} + 18 + \frac{1}{2} + 6 = 48.
\]
Suppose \( g(x) = \int_{1}^{x} t^2 \, dt \).

1. Calculate \( g(3) \)
2. Calculate \( g'(x) \)
3. Is \( f \) an increasing or decreasing function?
Example: Find area under the parabola \( y = x^2 + 1 \) from 0 to 2. 

\( x^2 + 1 > 0 \) for all \( x \) in \([0, 2]\). 

\( f(x) = x^2 + 1 \) is cts on \([0, 2]\) hence

\[
A = \int_{0}^{2} (x^2 + 1) \, dx = \left. \frac{x^3}{3} + x \right|_{0}^{2} = \left( \frac{2^3}{3} + 2 \right) - \left( \frac{0^3}{3} + 0 \right) = \frac{8}{3} + 2 = \frac{14}{3}.
\]

Example: Let \( g(x) = \int_{0}^{x} \sqrt{1 + t^2} \, dt \). Find \( g'(x) \).

Since \( f(x) = \sqrt{1 + x^2} \) is cts, then by FTC-C1

\[
g'(x) = \sqrt{1 + x^2}.
\]

Example: Find \( \frac{d}{dx} \int_{1}^{x^4} \sec t \, dt \).

Let \( u = x^4 \).

\[
\frac{d}{dx} \int_{1}^{x^4} \sec t \, dt = \frac{d}{dx} \int_{1}^{u} \sec t \, dt = \sec(u) \cdot (4x^3)
\]

Chain Rule

\[
= 4x^3 \cdot \sec(x^4)
\]
Suppose we have a particle moving along a straight line $x$ with position function $s(t)$, velocity $v(t)$ and acceleration $a(t)$.

Since $s'(t) = v(t)$

$$\int_{t_1}^{t_2} v(t) \, dt = s(t_2) - s(t_1)$$

Also $v'(t) = a(t)$

$$\int_{t_1}^{t_2} a(t) \, dt = v(t_2) - v(t_1)$$

**Example:**

$v(t) = t^2 - t - 6$ (m/sec)  \quad 1 \leq t \leq 4$

Find displacement:

$$\int_{1}^{4} (t^2 - t - 6) \, dt = \left[ \frac{t^3}{3} - \frac{t^2}{2} - 6t \right]_{1}^{4}$$

$$= -\frac{9}{2}$$

the particle moved $\frac{4.5}{2}$ m to the left.

Total distance traveled:

$$\int_{1}^{4} |v(t)| \, dt$$
Total distance $= \int_1^4 |t^2 - t - 6| \, dt$

$= \int_1^3 |t^2 - t - 6| \, dt + \int_3^4 |t^2 - t - 6| \, dt$

$= \int_1^3 (-t^2 + t + 6) \, dt + \int_3^4 (t^2 - t - 6) \, dt = \frac{61}{6}$ meters

$\frac{d}{dx} F(g(x)) = F'(g(x)) \cdot g'(x)$

$\Rightarrow \int F'(g(x)) \cdot g'(x) \, dx = F(g(x)) + C$

Let $u = g(x)$

$\int F'(g(x)) \cdot g'(x) \, dx = F(g(x)) + C = F(u) + C$

$= \int F'(u) \, du$

Then we have the following rule:

**The Substitution Rule:** If $u = g(x)$ is a staffe function whose range is an interval $I$ and $f$ is cts. on $I$, then

$$\int f(g(x)) \cdot g'(x) \, dx = \int f(u) \, du$$
\[ \int \sqrt{x-1} \, dx = \int u \, du \]
\[ = \frac{2}{3} u^{3/2} + C \]
\[ = \frac{2}{3} (x-1)^{3/2} + C \]

**Example:**

\[ \int x^3 \cos(x^4 + 2) \, dx = \frac{1}{4} \int \cos u \, du \]
\[ = \frac{1}{4} \sin u + C \]
\[ = \frac{1}{4} \sin(x^4 + 2) + C \]

\[ \int \cos^4 x \cdot \sin x \, dx \]
\[ = -\int u^4 \, du \]
\[ = -\frac{u^5}{5} + C \]
\[ = -\frac{\cos^5 x}{5} + C \]
\[ \int \frac{dx}{\sqrt{1-x^2}} = \int \frac{du}{u} \quad \text{where} \quad u = \sin^{-1} x \]
\[ du = \frac{1}{\sqrt{1-x^2}} \, dx \]
\[ = \ln |u| + C \]
\[ = \ln |\sin^{-1} x| + C \]

**Ex:**
\[ \int \frac{1 + x}{1 + x^2} \, dx \]
\[ = \int \frac{1}{1 + x^2} \, dx + \int \frac{x}{1 + x^2} \, dx \]

1. \[ u = \tan^{-1} x \]
\[ du = \frac{dx}{1 + x^2} \]
2. \[ u = 1 + x^2 \]
\[ du = 2x \, dx \]
\[ \frac{du}{2} = x \, dx \]

\[ = \int du = u + C = \tan^{-1} x + C \]
\[ = \int \frac{1}{2} \ln |u| + C_2 = \frac{1}{2} \ln (1 + x^2) + C_2 \]

\[ \Rightarrow \int \frac{1 + x}{1 + x^2} \, dx = \]
The Substitution Rule for Definite Integrals

If \( g' \) is continuous on \([a,b]\), and \( f \) is continuous on the range of \( g \), then

\[
\int_a^b f(g(x)) \, g'(x) \, dx = \int_{g(a)}^{g(b)} f(u) \, du
\]

Example: \( \int_0^4 \sqrt{3x+4} \, dx \)

\[
\begin{align*}
  u &= 3x + 4 \\
  x &= 0 \quad u = 4 \\
  x &= 4 \quad u = 16 \\

  \Rightarrow \quad \int_0^4 \sqrt{3x+4} \, dx &= \int_{4}^{16} \sqrt{u} \, \frac{du}{3} = \frac{1}{3} \int_{4}^{16} \sqrt{u} \, du \\

  &= \frac{1}{3} \left[ \frac{2}{3} u^{3/2} \right]_{4}^{16} \\

  &= \frac{2}{3} (16^{3/2} - 4^{3/2}) \\

  &= \frac{2}{3} (64 - 8) \\

  &= \frac{112}{9}
\end{align*}
\]
Example:

\[
\int_0^{\pi/4} \sin 4t \, dt
\]

\[
= \int_0^\pi \sin u \frac{du}{4} = \frac{1}{4} \int_0^\pi \sin u \, du
\]

\[
= -\frac{1}{4} \cos u \bigg|_0^\pi = -\frac{1}{4} (-1 - 1) = \frac{1}{2}
\]
Area between curves

\[ \int_a^b f(x) \, dx - \int_a^b g(x) \, dx = \int_a^b (f(x) - g(x)) \, dx \]

Ex. Find area of region bounded above by \( y = e^x \), below by \( y = x \) and held on the sides by \( x = 0 \) and \( x = 1 \).
Example 1: Find the area of the region bounded by the curves $y = \sqrt{x}$ and $y = x$.

Intersection points:

$\sqrt{x} = x$
$x = x^2$

$x^2 - x = 0$
$x(x-1) = 0$

$x = 0, x = 1$

$$A = \int_0^1 (\sqrt{x} - x) \, dx = \left. \frac{2}{3} x^{3/2} - \frac{x^2}{2} \right|_0^1 = \frac{2}{3} - \frac{1}{2} = \frac{1}{6}$$

Example 2: Find the area of the region between the curves $y = 9 - x^2$ and $y = x^2 + 1$ from $x = 0$ to $x = 3$.

Intersection points:

$9 - x^2 = x^2 + 1$
$8 = 2x^2$
$x = \pm 2$
$A = \int_0^2 (9-x^2)-(x^2+1) \, dx + \int_2^3 (x^2+1)-(9-x^2) \, dx$

$= \int_0^2 (8-2x^2) \, dx + \int_2^3 (2x^2-8) \, dx$

$= \left[ 8x - \frac{2}{3}x^3 \right]_0^2 + \left[ \frac{2}{3}x^3 - 8x \right]_2^3 = \ldots = \frac{46}{3}$

**Example 3:** Find the area of the region hided by the graphs of $\gamma^2 = x$ and $x - \gamma = 2$.

$x - \gamma = 2 \iff x = \gamma + 2$

$\Rightarrow \gamma^2 = x = \gamma + 2 \Rightarrow \gamma^2 - \gamma - 2 = 0$

$(\gamma+1)(\gamma-2) = 0$

$\gamma = -1$ or $\gamma = 2$

$x = 1 = (-1)^2$ or $x = (2)^2 = 4$
\[
\int_0^1 \left[ \sqrt{x} - (-\sqrt{x}) \right] \, dx + \int_1^4 \left[ \sqrt{x} - (x-2) \right] \, dx = \frac{9}{2} - \\
\begin{cases} 
X = y^2 \\
X = y + 2 
\end{cases}
\]

\[ A = \int_{-1}^{2} (y+2 - y^2) \, dy = \frac{y^2 + 2y - \frac{y^3}{3}}{2} \bigg|_{-1}^{2} = \\
= 2 + 4 - \frac{8}{3} - \left( \frac{1}{2} + 2(-1) - \frac{(-1)^3}{3} \right) = \\
= 6 - \frac{8}{3} - \frac{1}{2} + \frac{4}{3} = 8 - \frac{1}{2} - \frac{9}{3} = \\
= 5 - \frac{1}{2} = \frac{9}{2} - 
\]