Rolle's Theorem

Let \( f \) be a function s.t.

1. \( f \) is cont. on the closed interval \([a, b]\)
2. \( f \) is diff. on the open interval \((a, b)\)
3. \( f(a) = f(b) \)

Then there is a number \( c \) in \((a, b)\) s.t. \( f'(c) = 0 \)

Case 1: \( f(x) : k \Rightarrow f'(a) = 0 \)

Case 2: \( f(x) > f(a) \) for some \( x \in (a, b) \)

By extreme value thm, \( f \) has a max value in \([a, b]\)

\( f(a) \) or \( f(b) \) or max in \((a, b)\)

\( \Rightarrow \) by format thm, \( f'(c) = 0 \)

\( \Rightarrow \) \( f \) exists on \((a, b)\)

Case 3: \( f(x) < f(a) \)

Ex.

Show that the equation \( x^5 - 6x + 1 = 0 \) has at most one root in the interval \([-1, 1]\).

(Pf) Suppose \( x, y \in [-1, 1] \) : \( f(x) = f(y) \) \( \Rightarrow \) by Rolle's thm (check hypotheses)

\[ f'(c) = 0 \]

but \( f'(x) = 5x^4 - 6 = (5x^2 - 6)(x^2 + 6) = 0 \)

\[ x = \pm \sqrt{5/6} \approx 1.0456 \]

So we have a contradiction!
Remarks:

Ex:

\[ f \text{ cts. on } [a, b] \]
\[ f \text{ diff. on } (a, b) \]
\[ f(a) = f(b) \]
\[ \Rightarrow \exists c \in (a, b) : f'(c) = 0. \]

Where is \( c \)?

Ex:

\[ f(x) = |x|, \quad -1 \leq x \leq 1 \]

Ex:
Prove that the equation \( x^2 - x - 1 = 0 \) has exactly one real root.

First we use IVT to show that a root exists.

Let \( f(x) = x^3 + x - 1 \Rightarrow f(-1) < 0, f(1) > 0 \).

Since \( f \) is polynomial, it is cont. So by IVT there is \( c \in (0, 1) \) such that \( f(c) = 0 \).

Next we have to show this is the only one. Suppose \( c_1, c_2 \) such that \( f(c_1) = f(c_2) = 0 \).

By Rolle's thm \( \exists z \in (c_1, c_2) : f'(z) = 0 \).

But \( f'(x) = 3x^2 + 1 > 0 \forall x \in \mathbb{R} \), contradiction.

So there is only one solution of the equation.

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**Mean Value Theorem (MVT)** Let \( f \) be a cont'n s.th.

1. \( f \) is cont on closed interval \([a, b]\).
2. \( f \) is diff. on open interval \((a, b)\).

Then there is \( c \in (a, b) \) : \( f'(c) = \frac{f(b) - f(a)}{b - a} \)

or equiv.: \( f(b) - f(a) = f'(c)(b - a) \).
Apply Rolle's Thm to \( h(x) = f(x) - f(a) - \frac{f(b) - f(a)}{b - a}(x - a) \)

Example: \( f(x) = x^3 - x \quad [0, 2] \)

\( f'(x) = 3x^2 - 1 \)

\( f(2) = 6 \)

\( f(0) = 0 \)

\( 6 = f(2) - f(0) = f'(c)(2 - 0) = (3c^2 - 1)(2) = 6c^2 - 2 \)

\[ 6c^2 = 8 \quad c^2 = \frac{4}{3} \quad c = \pm \frac{2}{\sqrt{3}} \]

but \( c \in (0, 2) \) \( \quad c = \frac{2}{\sqrt{3}} \)
Example:

$f(c_1) = 10 \quad f'(x) \geq 2 \quad \text{for} \quad 1 \leq x \leq 4$

How small can $f(4)$ possibly be?

$f(4) - f(1) = f'(c) (4 - 1) \quad \text{for some } c \in (1, 4)$

$f(4) = f(1) + f'(c) (3) \quad = 10 + 3 f'(c) \quad \geq 10 + 3 (2) = 16$

$f(4) \geq 16$

So the smallest value for $f(4)$ is 16.

Theorem: If $f'(x) = 0$ for all $x \in (a, b)$ then $f$ is constant on $(a, b)$.

Corollary: If $f'(x) = g'(x)$ for all $x \in (a, b)$, then $f - g$ is constant on $(a, b)$; i.e., $f(x) = g(x) + c$ where $c$ is a constant.
A function is increasing on an interval if \( f(x_1) < f(x_2) \) whenever \( x_1 < x_2 \) in the interval. It is called decreasing if \( f(x_1) > f(x_2) \) for all \( x_1, x_2 \) in the interval.

**Test for monotonicity of functions:**

Suppose \( f \) is continuous on \([a, b]\) and differentiable on \((a, b)\).

1. If \( f'(x) > 0 \) for all \( x \) in \((a, b)\), then \( f \) is increasing on \([a, b]\).
2. If \( f'(x) < 0 \) for all \( x \) in \((a, b)\), then \( f \) is decreasing on \([a, b]\).
3. If \( f \) is differentiable on \([a, b]\) and \( f'(x_1) f'(x_2) < 0 \) for some \( x_1, x_2 \) in \((a, b)\), then \( f \) has a local maximum or minimum at \( c \) in \((a, b)\).

**Example 1:** Find the interval on which \( f(x) = e^{x^2} - x^4 \) is increasing or decreasing.

\[
f'(x) = 4x - 4x^3 = 4x(1-x^2) > 0
\]

\( x > 0 \)

\(-1 < x < 1\)

\( x < 0 \)

If \( f \) has a local max or min, then \( c \) is a critical number. We also saw that the curve is not nice - so we need a test that will tell us whether or not \( f \) has local extremum at a critical number.

**The 1st derivative test:** Suppose \( c \) is a critical number of a continuous function \( f \).

a) If \( f' \) changes from + to - at \( c \), then \( f \) has local maximum at \( c \).

b) If \( f' \) changes from - to + at \( c \), then \( f \) has local minimum at \( c \).

c) If \( f' \) does not change sign, then \( f \) has no local extremum at \( c \).
Example: (cont'd)

(b) Find the local max and min values of \( f \):

\[
f(-1) = 2(-1)^2 - (-1)^4 = 1 \quad \text{local max}
\]
\[
f(0) = 0 \quad \text{local min}
\]
\[
f(1) = 1 \quad \text{local max}
\]

(c) Sketch graph of \( f \):

\[
f(x) = 2x^2 - x^4 = x^2(2-x^2) = 0 \quad x=0
\]
\[
x^2-2 = 0 \quad x = \pm \sqrt{2}
\]

Example:
Find the local and absolute extreme values of the function \( f(x) = x + \frac{1}{x} \) on \( \frac{1}{2} \leq x \leq 3 \) and sketch the graph.
Critical numbers: \( x = +1, x = -1, x = 0 \)

\[
f(x) = \frac{x + \frac{1}{x}}{x^2} = \frac{x^2 - 1}{x^2}
\]

\[
f'(x) = 1 - \frac{1}{x^2} = \frac{x^2 - 1}{x^2}
\]

\[
f\left(\frac{1}{2}\right) = \frac{1}{2} + 2 = \frac{5}{2}
\]

\[
f(-1) = -1 + \frac{1}{-1} = -2
\]

\[
f(3) = 3 + \frac{1}{3} = \frac{10}{3}
\]

\[
f(1) = 1 + 1 = 2 \quad \text{absolute min}
\]

\[
\lim_{x \to 0^+} \frac{x^2 + 1}{x} = \frac{1}{0^+} = \infty
\]

\[
f'(x) = \frac{x^2 + 1}{x^2} = 0 \quad \text{never}
\]

\[
\text{as } x \to 2^+ \quad f'(x) = -\infty
\]

\[
\text{as } x \to 2^- \quad f'(x) = +\infty
\]

\[
f(x) = \text{odd} \quad f(-x) = -f(x)
\]

\[
f(x) \quad \text{loc max}
\]

\[
f(x) \quad \text{loc min}
\]
Show that \( \frac{a + \frac{1}{a}}{b + \frac{1}{b}} \) whenever \( 1 \leq a \leq b \).

Let \( f(x) = x + \frac{1}{x} \). We just saw that \( f \) is increasing on \((1, \infty)\). Then if \( 1 \leq c < b \), then

\[
f(c) = c + \frac{1}{c} < f(b) = b + \frac{1}{b}.
\]

What does \( f'' \) say about \( f \)?

**Concavity and Points of Inflection.**

**Def.** If the graph of \( f \) lies above all of its tangents on an interval \( I \), then \( f \) is concave upward on \( I \).

If the graph of \( f \) lies below all of its tangents on \( I \), then it is called concave downward on \( I \).

Slopes increasing

Slopes decreasing

**Test for Concavity.** Suppose \( f \) is diff. twice on \( I \).

1. If \( f''(x) > 0 \) for all \( x \in I \), then \( f \) is concave upward on \( I \).

2. If \( f''(x) < 0 \) for all \( x \in I \), then \( f \) is concave downward on \( I \).
Proof: In concave, \( f''(x) > 0 \Rightarrow f' \) is increasing.

**Definition:** A point \( P \) on a curve is called a point of inflection if the curve changes from concave upward to concave downward or vice versa.

**Example:** Let \( f(x) = x^3 - x \)

(c) Find intervals of concavity

(d) \( x \) - coordinates of points of inflection

\[
\begin{align*}
  f'(x) &= 3x^2 - 1 \\
  f''(x) &= 6x \\
  f''(x) &> 0 & x > 0
\end{align*}
\]

concave downward \( \searrow \) concave upward

\( 0 \) \( \bigcirc \) pt. of inflection
Another application of 2nd derivative is in finding max and min.

2nd derivative Test:

Suppose \( f'' \) is continuous on an open interval containing \( c \).

1. If \( f'(c) = 0 \) and \( f''(c) > 0 \) \( \Rightarrow \) \( c \) local minimum
2. If \( f'(c) = 0 \) and \( f''(c) < 0 \) \( \Rightarrow \) \( c \) local maximum

Example: Discuss the curve \( y = x^4 - 4x^3 \) with respect to concavity, pts of inflection and local extremes. Use this info to sketch the curve.

\[ f(x) = x^4 - 4x^3 \]
\[ f'(x) = 4x^3 - 12x^2 = 4x^2(x - 3) \]
\[ f''(x) = 12x^2 - 24x = 12x(x - 2) \]

Critical #5:

\[ f''(x) = 0 \Rightarrow x = 0, x = 3 \]
\[ f''(0) = 0 \quad ? \]
\[ f''(3) = 36 > 0 \Rightarrow 3 \text{ is local minimum} \]

Use different criterion:

\[ f(3) = 3^4 - 4 
\[ = 81 - 48 \]
\[ = 33 \]
\[ f''(x) = 12x(x-2) = 0 \]

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\[ 0 \quad \text{pts. of inflection} \quad 2 \]

Graph:

\[ f(0) = 0 \quad f(2) = 2^4 - 4 \cdot 2^3 = 2^4 \cdot (1 - 2) = -16 \]

\[ f \text{ has absolute min at } x = 2 \]
Example:

\[ f(0) > 0 \]

\[ f'(x) = 0 \quad f'(2) = -1 \] for all \( x \) in the interval (0, 2)

\[ f''(x) < 0 \quad x < 2 \]

\[ f''(x) > 0 \quad x > 2 \]

\[ f''(x) < 0 \quad 0 < x < 1 \] or \( x > 4 \)

\[ f''(x) > 0 \quad 1 < x < 4 \]

\[ \lim_{{x \to \infty}} f(x) = 1 \]

\[ f(-x) = f(x) \quad \forall x \]

It's enough to draw the graph.

![Graph of f(x) with critical points and asymptote.](image)

\[ f(2) = -1 \] local min.

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Reflection points:

\( y = 1 \) Horizontal Asymptote