So far we used $\frac{dy}{dx}$ to denote the derivative of $y$ wrt $x$. We have regarded this as a single entity and not a ratio. Here we are going to give $dy$ and $dx$ separate meanings in such a way that their ratio is equal to the derivative.

**Def.** Let $y = f(x)$, where $f$ is diff. ft. Then the differential $dx$ is an indep. variable, that is $dx$ can be given the value of any real number.

The differential $dy$ is then given by

$$dy = f'(x) \, dx$$

**Ex.**

(a) Find $dy$ if $y = \cos x$

(b) Find value of $dy$ when $x = \frac{\pi}{6}$, $dx = 0.05$

(a) $dy = -\sin x \, dx$

(b) $x = \frac{\pi}{6}$, $\sin x = \sin \frac{\pi}{6} = \frac{1}{2}$

$dx = 0.05$

$$dy = (-\sin \, \frac{\pi}{6})(0.05) = \left(-\frac{1}{2}\right)(0.05) = -0.025$$
The geometric meaning:

\[ \Delta y = f(x + \Delta x) - f(x) \]

\[ \frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} \]

\[ \frac{\Delta y}{\Delta x} \approx \frac{dy}{dx} \text{ for } \Delta x \text{ small.} \]

If we choose \( \Delta x = \Delta x \), then \( \Delta y \approx dy \).

This is used to find approximate values of functions.

Suppose \( f(e) \) is known, and we want to calculate value of \( f(e + \Delta x) \) where \( \Delta x \) is small. Since

\[ f(e + \Delta x) = f(e) + dy \]

we have

\[ f(e + \Delta x) \approx f(e) + dy \]
Example: The demand function for a product is given by

\[ p = f(q) = 20 - \sqrt{q} \]

where \( p \) is the price per unit in dollars for \( q \) units. By using the slope tool, approximate the price when 99 units are demanded.

We want to approximate \( f(99) \). We know:

\[ f(q + \Delta q) \approx f(q) + dp \]

where

\[ dp = f'(q) \Delta q = -\frac{\Delta q}{2\sqrt{q}} \]

\( q_0 = 100 - 1 \) \( \implies \) Take \( \Delta q = 9 = 10 \)

\( \implies f(99) \approx f(100) - \frac{\Delta q}{2\sqrt{100}} = f(100) + \frac{1}{2 \cdot 10} \)

\[ = \sqrt{10} + \frac{1}{20} = 10 + 0.05 = 10.05 \]

Example: P. 169 #25:

The edge of a cube was found to be 30 cm with a possible error in measurement of 0.1 cm. Use differentials to estimate the maximum possible error in computing the volume of the cube.

\[ V = x^3 \]

\[ dV = 3x^2 \, dx \]

\[ x = 30 \]

\[ dx = 0.1 \]

\[ - 3 \times (30)^2 (0.1) = 270 \, \text{cm}^3 \]
This seems a huge error. However a better factor is given by the relative error

\[ \frac{dV}{V} = \frac{270}{(30)^3} = \frac{270}{27,000} = \frac{1}{100} = 0.01 \]

Relative error = \( \frac{1}{100} \) – \( \frac{1}{100} \) \( \text{percentage error} \)

**Linear Approximation:**

We have

\[ f(x+\Delta x) \approx f(x) + df \]

\[ = f(x) + f'(x) \Delta x \]

\[ x = a + \Delta x \]

\[ f(x) \approx f(a) + f'(a)(x-a) \] ← equation of tangent at point \((a, f(a)))\)

This means that we are approximating the curve \( y = f(x) \) with the tangent line.

\[ L(x) = f(a) + f'(a)(x-a) \]

is the linearization of \( f \) at \( a \).
Example

Find the linearization of \( f(x) = x^3 \) at \( a = 1 \):

\[
L(x) = f(a) + f'(a)(x-a)
\]

\[
= (1)^3 + (3x^2)\bigg|_{x=1}
\]

\[
= 1 + 3(x-1)
\]

\[
= 3x - 2
\]

So \( x^3 \approx 3x - 2 \)

The linearization is the best first degree (linear) approximation to \( f(x) \) near \( a \).

For a better approximation, we consider 2nd degree (quadratic) approximation \( P(x) \).

To make sure the approximation is good, we want:

1. \( P(a) = f(a) \)
2. \( P'(a) = f'(a) \)
3. \( P''(a) = f''(a) \)
Example: Find a polynomial approximation of the function $f(x) = \cos x$ near $0$.

Let $P(x) = a + bx + cx^2$

$P'(x) = b + 2cx$

$P''(x) = 2c$

Then

$\begin{align*}
P(0) &= f(0) = 1 \\
P'(0) &= f'(0) = -\sin 0 = 0 \\
P''(0) &= f''(0) = -\cos 0 = -1
\end{align*}$

Solving the system of equations:

$\begin{align*}
a &= 1 \\
b &= 0 \\
c &= -\frac{1}{2}
\end{align*}$

Hence $P(x) = 1 - \frac{1}{2}x^2$.

In general, if we consider

$P(x) = A + B(x-a) + C(x-a)^2$

$P(a) = f(a)$

$P'(a) = f'(a)$

$P''(a) = f''(a)$

Solving for $A$, $B$, and $C$,

$A = f(a)$

$B = f'(a)$

$2C = f''(a)$

Hence

$P(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2$

is the polynomial approximation of $f(x)$ close to $a$. 

7. \( f(x) = \sqrt[3]{1-x} = (1-x)^{1/3} \Rightarrow f'(x) = -\frac{1}{3}(1-x)^{-2/3} \), so \( f(0) = 1 \) and \( f'(0) = -\frac{1}{3} \). Thus, \( f(x) \approx f(0) + f'(0)(x-0) = 1 - \frac{1}{3}x \). We need \( \sqrt[3]{1-x} - 0.1 < 1 - \frac{1}{3}x < \sqrt[3]{1-x} + 0.1 \), which is true when \(-1.204 < x < 0.706\).

8. \( f(x) = \tan x \Rightarrow f'(x) = \sec^2 x \), so \( f(0) = 0 \) and \( f'(0) = 1 \).

Thus, \( f(x) \approx f(0) + f'(0)(x-0) = 0 + 1(x-0) = x \).

We need \( \tan x - 0.1 < x < \tan x + 0.1 \), which is true when \(-0.63 < x < 0.63\).

9. \( f(x) = \frac{1}{(1+2x)^4} = (1+2x)^{-4} \Rightarrow f'(x) = -4(1+2x)^{-5}(2) = \frac{-8}{(1+2x)^5} \), so \( f(0) = 1 \) and \( f'(0) = -8 \).

Thus, \( f(x) \approx f(0) + f'(0)(x-0) = 1 + (-8)(x-0) = 1 - 8x \).

We need \( \frac{1}{(1+2x)^4} - 0.1 < 1 - 8x < \frac{1}{(1+2x)^4} + 0.1 \), which is true when \(-0.045 < x < 0.055\).

10. \( f(x) = e^x \Rightarrow f'(x) = e^x \), so \( f(0) = 1 \) and \( f'(0) = 1 \).

Thus, \( f(x) \approx f(0) + f'(0)(x-0) = 1 + 1(x-0) = 1 + x \).

We need \( e^x - 0.1 < 1 + x < e^x + 0.1 \), which is true when \(-0.483 < x < 0.416\).