COLUMBIA UNIVERSITY

CALCULUS I (Math S1101X(3))

Final Exam – August 13, 2009

Instructor: Dr. Sandro Fusco

Family Name: ________________________________

Given Name: ________________________________

Instructions:

1. You have 3 hours.
2. Your work must justify the answer you give.
3. There are 10 questions (Problems 1-10) plus 3 bonus questions (Problems 11-13).
4. Attempt the first ten questions first, and then work on the bonus questions.
5. No calculators, lecture notes and/or books are permitted.
6. Point values are as shown.
7. This is the first of fifteen (15) pages.

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Problem 1: (10 Points)

Let \( f(x) = x^2 - 8 \), and \( g(x) = \frac{1}{\sqrt{x-1}} \).

1. Find the composition function \( g \circ f \), and its domain.

\[
(g \circ f)(x) = g(f(x)) = g(x^2 - 8) = \frac{1}{\sqrt{x^2 - 8}} = \frac{1}{\sqrt{x^2 - 9}}
\]

\( \text{dom}(f) = \mathbb{R} \)

\( \text{dom}(g) = \{ x \in \mathbb{R} : x - 1 > 0 \} = \{ x \in \mathbb{R} : x > 1 \} \)

\( \text{dom}(g \circ f) = \{ x \in \text{dom}(f) : f(x) \in \text{dom}(g) \} \)

\( = \{ x \in \mathbb{R} : x^2 - 8 > 1 \} \)

\( = \{ x \in \mathbb{R} : x^2 - 9 > 0 \} = (-\infty, -3) \cup (3, +\infty) \)

2. Find the composition function \( f \circ g \), and its domain.

\[
f \circ g (x) = f(g(x)) = f \left( \frac{1}{\sqrt{x-1}} \right) = \frac{1}{x-1} - 8 = \frac{1 - 8(x-1)}{x-1} = \frac{-8x+9}{x-1}
\]

\( \text{dom}(f \circ g) = \{ x \in \text{dom}(g) : g(x) \in \text{dom}(f) \} \)

\( = \{ x > 1 : \frac{1}{\sqrt{x-1}} \in \mathbb{R} \} \)

\( = \{ x > 1 \} = (1, +\infty) \)
Problem 2: (10 Points)

Use the Intermediate Value Theorem to show that there is a root of the equation

\[ 2x^3 + x^2 + 2 = 0 \]

in the interval (-2, -1).

Let \( f(x) = 2x^3 + x^2 + 2 \)

Since \( f \) is a polynomial, we know that \( f \) is continuous everywhere, hence on \([-2, -1]\)

\[ f(-2) = 2(-2)^3 + (-2)^2 + 2 = -10 \]
\[ f(-1) = 2(-1)^3 + (-1)^2 + 2 = 1 \]

Hence by IVT, there is a \( c \in (-2, -1) \) such that \( f(c) = 0 \)

Such a \( c \) is a root of the equation \( 2x^3 + x^2 + 2 = 0 \).
Problem 3: (10 Points)

Let

\[ f(x) = \begin{cases} 
2x - 4 & \text{if } x > 2 \\
1 & \text{if } x = 2 \\
2 - x & \text{if } x < 2 
\end{cases} \]

1. Sketch the graph of \( f \) and find its domain, and range.

\[ \text{dom}(f) = \mathbb{R} \]
\[ \text{range}(f) = (0, +\infty) \]

2. Does \( \lim_{x \to 2} f(x) \) exist? [Hint: Find \( \lim_{x \to 2^-} f(x) \) and \( \lim_{x \to 2^+} f(x) \)]

\[ \lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} (2 - x) = 0 \]
\[ \lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} (2x - 4) = 0 \]

Hence \( \lim_{x \to 2} f(x) \) exists and equals 0.

3. On which interval(s) is \( f \) continuous?

\( f \) is cts. for \( x < 2 \) as \( f(x) = 2 - x \) is a polynomial.
\( f \) is cts. for \( x > 2 \) as \( f(x) = 2x - 4 \) is a polynomial.

At \( x = 2 \)
\[ \lim_{x \to 2} f(x) = 0 \neq f(2) = 1 \rightarrow f \text{ is not cts. at } 2. \]

Hence \( f \) is cts. on \( (-\infty, 2) \cup (2, +\infty) \).
Problem 4: (10 Points)

Compute the derivatives of the following functions. You don't have to simplify your answer.

1. \( f(x) = \cos^2 x \)

\[
f'(x) = 2 \cos x (-\sin x) = -2 \sin x \cos x \quad \text{chain rule}
\]

2. \( f(x) = (5x^3 + 4)^2 (3x + 1) \)

\[
f''(x) = \left[ (5x^3 + 4)^2 \right]' (3x + 1) + (5x^3 + 4)^2 (3x + 1)' = 2(5x^3 + 4)(15x^2)(3x + 1) + 3(5x^3 + 4)^2 \quad \text{chain rule - product rule}
\]

3. \( f(x) = \sqrt{x^2 + 1} \)

\[
f'(x) = \frac{1}{2\sqrt{x^2 + 1}} \cdot (x^2 + 1)' = \frac{2x}{2\sqrt{x^2 + 1}} = \frac{x}{\sqrt{x^2 + 1}} \quad \text{chain rule - power rule}
\]

4. \( f(x) = \frac{x}{x + 1} \)

\[
f''(x) = \frac{x'(x+1) - x(x+1)'}{(x+1)^2} = \frac{1(x+1) - x(1)}{(x+1)^2} = \frac{x+1 - x}{(x+1)^2} = \frac{1}{(x+1)^2} \quad \text{quotient rule}
\]
Problem 5: (10 Points)

A point moves along the parabola \( y = x^2 \) so that \( \frac{dy}{dt} = 2 \). Compute \( \frac{dx}{dt} \) of the point, as it passes the point (2,4).

\[
y = x^2
\]

\[
\frac{dy}{dt} = \frac{d}{dt}(x^2) = 2x \frac{dx}{dt}
\]

When the point passes the point (2,4) we know that \( x = 2 \); hence we have

\[
2 = \frac{dy}{dt} = 2(2) \frac{dx}{dt}
\]

\[
\Rightarrow \quad \frac{dx}{dt} = \frac{1}{2}
\]
Problem 6: (10 Points)

Find two positive numbers whose product is 100 and whose sum is a minimum.

Let \( x, y > 0 \) be two positive real numbers such that \( xy = 100 \).

We want to minimize \( x + y = x + \frac{100}{x} \).

Let \( f(x) = x + \frac{100}{x} = \frac{x^2 + 100}{x}, \quad x > 0 \).

Then \( f'(x) = 1 - \frac{100}{x^2} = \frac{x^2 - 100}{x^2} \).

Critical numbers are \( x = 0, \circ (x = 10), \circ (x = -10) \) not in the domain.

So \( f \) has a local minimum at \( x = 10 \).

Also, since \( f'(x) < 0 \) for \( x < 10 \), and \( f'(x) > 0 \) for \( x > 10 \), we have that \( x = 10 \) is an absolute minimum by 1st Der. Test for absolute extreme (if it is CTS on its domain or if it is a rational function).

Thus, the two numbers we are looking for are \( x = 10, y = 10 \).
Problem 7: (10 Points)

Draw the graph of the function $f$ satisfying:

- $f(-2)$ does not exist
- $\lim_{x \to -2^+} f(x) = +\infty$ and $\lim_{x \to -2^-} f(x) = -\infty$
- $\lim_{x \to +\infty} f(x) = 1$ and $\lim_{x \to -\infty} f(x) = 0$
- $f'(x) < 0$ for $-\infty < x < -2$ and $-2 < x < \infty$

\[
\begin{align*}
& x = -2 \text{ is a vertical asymptote} \\
& y = 1 \text{ are horizontal asymptotes} \\
& y = 0 \text{ decreasing everywhere}
\end{align*}
\]
Problem 8: (10 Points)

Find the function \( f \) for which \( f''(x) = x - 1, \ f'(0) = 2, \) and \( f(1) = -1/3. \)

\[
f''(x) = x - 1
\]

\[
f'(x) = \frac{x^2}{2} - x + C
\]

\[
f(x) = \frac{x^3}{6} - \frac{x^2}{2} + Cx + D
\]

\[
f(0) = 2 \quad \Rightarrow \quad D = 2
\]

\[
f(1) = \frac{-1}{3} \quad \Rightarrow \quad \frac{1}{6} - \frac{1}{2} + C + 2 = \frac{-1}{3} \quad \Rightarrow \quad C = -2
\]

\[
\Rightarrow \quad f(x) = \frac{x^3}{6} - \frac{x^2}{2} - 2x + 2
\]
Problem 9: (10 Points)

Compute the following integrals, using any method you like.

1. \[ \int \frac{\sqrt{3}+5x}{x^4} \, dx \]
   
   \[
   = \left. \frac{x^4 + 5x^2}{2} \right|_1^2 = \frac{2^4 + 5(2)^2}{2} - \left( \frac{1 + 5}{2} \right) \\
   = 4 + 10 - \frac{1}{4} - \frac{5}{2} = 14 - \frac{11}{4} \\
   = \frac{45}{4} = 11 \frac{1}{4} 
   \]

2. \[ \int \frac{\cos \sqrt{x}}{\sqrt{x}} \, dx \]
   
   \[ \text{Let } u = \sqrt{x}, \quad du = \frac{1}{2\sqrt{x}} \, dx \]
   
   \[ = 2 \int \cos(u) \, du \]
   
   \[ = 2 \sin(u) + C = 2 \sin(\sqrt{x}) + C \]

3. \[ \int \frac{1}{x^2} \sqrt{1 + \frac{1}{x}} \, dx \]
   
   \[ \text{Let } u = 1 + \frac{1}{x}, \quad du = -\frac{1}{x^2} \, dx \]
   
   \[ = -\int \frac{1}{\sqrt{u}} \, du \]
   
   \[ = \int_2^4 \sqrt{u} \, du = \frac{2}{3} u^{3/2} \bigg|_2^4 = \frac{2}{3} \left( 4^{3/2} - 2^{3/2} \right) \]
   
   \[ = \frac{2}{3} \left( 8 - 2\sqrt{2} \right) = \frac{4}{3} \left( 4 - \sqrt{2} \right) \]

4. \[ \int (x + \sin x) \, dx \]
   
   \[ = \frac{x^2}{2} - \cos x + C \]
Problem 10: (10 Points)

Find the area of the region bounded by the curves $y = \sqrt{x}$ and $y = x$. Draw the picture of this region.

\[ \sqrt{x} = x \implies x = x^2 \implies x^2 - x = 0 \implies x(x-1) = 0 \]

So $x=0$ and $x=1$ are the $x$-coordinates where the two curves intersect.

Also $\sqrt{x} > x$ for $0 < x < 1$. Hence

\[ A = \int_0^1 (\sqrt{x} - x) \, dx = \frac{2}{3} x^{3/2} - \frac{x^2}{2} \bigg|_0^1 \]

\[ = \frac{2}{3} - \frac{1}{2} = \frac{1}{6} \]
Problem 11: (Bonus - 10 Extra Points)

Show that \( \frac{1}{a} < \frac{1}{b} \) whenever \( 1 < a < b \)

Let \( f(x) = x + \frac{1}{x} \) with \( x \neq 0 \).

\[
f'(x) = 1 - \frac{1}{x^2} = \frac{x^2 - 1}{x^2} = \frac{(x+1)(x-1)}{x^2}.
\]

Critical numbers are \( x = 0 \), \( x = \pm 1 \).

Not in the domain of \( f \).

\[
\begin{array}{cccccc}
  x^2 - 1 > 0 & + & - & - & + \\
  x^2 > 0 & - & - & + & - \\
  f' & & & & \\
\end{array}
\]

Hence \( f' > 0 \) for \( x > 1 \), which implies that \( f \) is increasing for \( x > 1 \); i.e. if \( 1 < a < b \)

Then \( f(a) < f(b) \).

\[
\begin{array}{ccc}
  a + \frac{1}{a} & < & b + \frac{1}{b} \\
\end{array}
\]
Problem 12: (BONUS - 10 Extra Points)

Compute the following integral, using any method you like.

\[
\int \frac{1+x}{1+x^2} \, dx = \int \frac{dx}{1+x^2} + \int \frac{x}{1+x^2} \, dx
\]

\(\text{\(1\)}\)

\[
\int \frac{dx}{1+x^2} = \int du = u + c_1, \quad \text{Let} \ u = \arctan x
\]

\[
du = \frac{1}{1+x^2} \, dx
\]

\[\arctan x + c_1\]

\(\text{\(2\)}\)

\[
\int \frac{x}{1+x^2} \, dx
\]

\[
= \frac{1}{2} \int \frac{1}{u} \, du \quad \text{Let} \ u = 1+x^2
\]

\[
du = 2x \, dx
\]

\[\frac{1}{2} \ln |u| + c_2 = \frac{1}{2} \ln (1+x^2) + c_2\]

\[\Rightarrow \int \frac{1+x}{1+x^2} \, dx = (1) + (2) = \arctan x + \frac{1}{2} \ln (1+x^2) + C\]
Problem 11: (BONUS - 10 Extra Points)

Below is the graph of $f''$, the derivative of $f$.

1. At what $x$-values does $f$ have local minima? At what $x$-values does $f$ have local maxima?

\[
\begin{array}{c|c|c|c}
 x & f' & f'' & \text{Loc. Min/Max} \\
-2 & - & + & \text{Loc. min at } x = -2 \\
2 & + & - & \text{Loc. max at } x = 5 \\
5 & - & + & \\
\end{array}
\]

2. Where is $f$ concave up?

\[
\begin{array}{c|c|c|c|c}
 x & f'' & \text{Concavity} \\
-1 & + & CD \\
2 & - & CU \\
4 & + & CD \\
\end{array}
\]

$f$ is concave up on $(-\infty, -1) \cup (2, 4)$.
3. At which $x$-values does $f$ have a horizontal tangent line?

$f$ has horizontal tangent line when $f'(x) = 0$

hence at $x = -2, 2, 5$

4. Calculate $\int f''(x) \, dx$.

\[
\int_{-1}^{2} f''(x) \, dx = -\int_{-1}^{2} f''(x) \, dx
\]

by FTC.

\[
= -f'(1) \bigg|_{-1}^{2}
\]

\[
= -f'(2) + f'(-1)
\]

\[
= -0 + 4
\]

\[
= 4
\]

5. What is the sign of the slope of the tangent line to $f(x)$ at $x = 1$?

$f'(1) > 0$  \(\Rightarrow\)  slope of the tangent line at $x = 1$ is positive

Enjoy the rest of the summer!