# Solutions to Final Exam
Calculus 1, Section 012
December 22, 2009

<table>
<thead>
<tr>
<th>#</th>
<th>Topic</th>
<th>Points</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Computing Integrals</td>
<td>/20 points</td>
</tr>
<tr>
<td>2</td>
<td>Computing Limits</td>
<td>/20 points</td>
</tr>
<tr>
<td>3</td>
<td>Continuity, Differentiability, Integrability</td>
<td>/16 points</td>
</tr>
<tr>
<td>4</td>
<td>Derivatives by Definition</td>
<td>/10 points</td>
</tr>
<tr>
<td>5</td>
<td>Implicit Differentiation</td>
<td>/21 points</td>
</tr>
<tr>
<td>6</td>
<td>Derivatives and Integrals</td>
<td>/18 points</td>
</tr>
<tr>
<td>7</td>
<td>Volume</td>
<td>/25 points</td>
</tr>
<tr>
<td>8</td>
<td>Curve Sketching, Antiderivatives</td>
<td>/18 points</td>
</tr>
<tr>
<td>9</td>
<td>Curve Identification</td>
<td>/8 points</td>
</tr>
<tr>
<td>10</td>
<td>Riemann Sums, Integrability</td>
<td>/24 points</td>
</tr>
<tr>
<td>11</td>
<td>Optimization</td>
<td>/20 points</td>
</tr>
<tr>
<td></td>
<td><strong>Total</strong></td>
<td><strong>/200 points</strong></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>Extra Credit</th>
<th>Points</th>
</tr>
</thead>
<tbody>
<tr>
<td>EC1</td>
<td></td>
<td>/10 points</td>
</tr>
<tr>
<td>EC2</td>
<td></td>
<td>/10 points</td>
</tr>
<tr>
<td></td>
<td><strong>Total + EC</strong></td>
<td><strong>/220 points</strong></td>
</tr>
</tbody>
</table>
1. Compute the following integrals. [10 points each = 20 points]

(a) \[ \int_0^b x \sqrt{b^2 - x^2} \, dx \]

**Solution:** Use the substitution \( u = b^2 - x^2 \). Since this integral is with respect to \( x \), the \( b \) is to be treated as a constant. Therefore, \( du = -2x \, dx \). Since this is a definite integral, we either have to convert to an indefinite integral and do that first, or we have to change the limits of integration. To change the limits of integration, calculate as follows: when \( x = 0 \), we have \( u = b^2 \), and when \( x = b \), we have \( u = 0 \). So the \( u \)-version of this integral is \( \int_{b^2}^0 -\frac{1}{2} \sqrt{u} \, du \). Equivalently, it’s \( \frac{1}{2} \int_0^{b^2} \sqrt{u} \, du \). Now calculate as usual.

\[
\frac{1}{2} \int_0^{b^2} \sqrt{u} \, du = \frac{1}{2} \cdot \frac{2}{3} u^{3/2} \bigg|_0^{b^2} = \frac{1}{3} (b^2)^{3/2} = \frac{1}{3} b^3
\]

If you don’t want to change the limits of integration, substitute into the indefinite integral instead, integrate the \( u \)-integral, then substitute \( x \) back in.

\[
\int x \sqrt{b^2 - x^2} \, dx = \int -\frac{1}{2} \sqrt{u} \, du = -\frac{1}{2} \cdot \frac{2}{3} u^{3/2} = -\frac{1}{3} (b^2 - x^2)^{3/2}.
\]

Finally, plug the original limits of integration into the answer for \( x \):

\[
-\frac{1}{3} (b^2 - b^2)^{3/2} - \left( -\frac{1}{3} (b^2 - 0^2)^{3/2} \right) = \frac{1}{3} b^3.
\]

(b) \[ \int \frac{e^\frac{x}{x}}{x^2} \, dx \]

**Solution:** Use \( u \)-substitution with \( u = \frac{x}{x} \). Then \( du = \frac{x}{x^2} \, dx \). The \( u \)-version of the original integral is \( \frac{1}{\pi} \int e^u \, du \), so we compute

\[
\int \frac{e^\frac{x}{x}}{x^2} \, dx = -\frac{1}{\pi} \int e^u \, du = -\frac{1}{\pi} e^u + C = -\frac{1}{\pi} e^{\frac{x}{x}} + C.
\]
2. Compute the following limits. [10 points each = 20 points]

(a) \( \lim_{x \to \infty} x^3 e^{-x} \)

**Solution:** This is an indeterminate limit of the form \( \infty \cdot 0 \). Convert it to a quotient, then apply L'Hôpital's Rule to the resulting indeterminate limit of form \( \frac{\infty}{\infty} \). L'Hôpital's Rule produces another \( \frac{\infty}{\infty} \) limit, so apply again, and then again, and then you're done.

\[
\lim_{x \to \infty} x^3 e^{-x} = \lim_{x \to \infty} \frac{x^3}{e^x} = \lim_{x \to \infty} \frac{3x^2}{e^x} = \lim_{x \to \infty} \frac{6x}{e^x} = \lim_{x \to \infty} \frac{6}{e^x} = 0
\]

(b) \( \lim_{x \to 1^+} \frac{x}{x - 1} - \frac{1}{\ln x} \)

**Solution:** This is an indeterminate limit of the form \( \infty - \infty \). Convert it to a quotient by making a common denominator. The result is a \( \frac{0}{0} \) limit, so apply L'Hôpital's Rule. Repeat as necessary.

\[
\lim_{x \to 1^+} \frac{x}{x - 1} - \frac{1}{\ln x} = \lim_{x \to 1^+} \frac{x \ln x - x + 1}{(x - 1) \ln x}
\]

\[
= \lim_{x \to 1^+} \frac{\ln x + x/x - 1}{\ln x + (x - 1)/x}
\]

\[
= \lim_{x \to 1^+} \frac{x \ln x}{x \ln x + x - 1}
\]

\[
= \lim_{x \to 1^+} \frac{\ln x + x/x}{\ln x + x/x + 1}
\]

\[
= \lim_{x \to 1^+} \frac{\ln x + 1}{\ln x + 2}
\]

\[
= \frac{1}{2}
\]

3. This question is about the relationships between three properties of functions that we studied this semester: continuity, differentiability, and integrability. For each part, give an example of a function (a graph or a formula) that satisfies the properties, or explain why no such function exists. [4 points each = 16 points]

(a) Continuous and not differentiable.

**Solution:** A function with a sharp bend or cusp can be continuous and not differentiable. The standard example is \( f(x) = |x| \), but there are many others.

(b) Discontinuous and integrable.

**Solution:** A function is integrable if it is continuous, but is also integrable if it has only finitely many jump discontinuities. One example is a step function, but any example with finitely many jump discontinuities will do.

(c) Integrable and not differentiable.

**Solution:** The step function, or any function with only finitely many jump discontinuities will be integrable, as above, but cannot be differentiable because all differentiable functions are continuous.

(d) Differentiable and not integrable.

**Solution:** No such function exists. If a function is differentiable, then it is continuous and if it is continuous, then it is integrable. Therefore, differentiable implies integrable.
4. Let \( f(x) = 3x^2 - x \). Use the limit definition of the derivative to show that \( f'(x) = 6x - 1 \).

**Solution:** Recall that the limit definition of the derivative is

\[
 f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.
\]

Applying that to this problem, we calculate as follows.

\[
 f'(x) = \lim_{h \to 0} \frac{3(x+h)^2 - (x+h) - 3x^2 + x}{h} = \lim_{h \to 0} \frac{3x^2 + 6xh + 3h^2 - x - h - 3x^2 + x}{h} = \lim_{h \to 0} \frac{6xh + 3h^2 - h}{h} = \lim_{h \to 0} 6x + 3h - 1 = 6x - 1
\]

5. Throughout this problem, consider the curve \( y^2 = x^3(2-x) \).

(a) Find an equation for the tangent line to this at the point \((1, 1)\). [8 points]

**Solution:** To find the slope of the tangent line to this curve, you need to find the derivative of \( y \) with respect to \( x \). You could solve for \( y \) by taking a square root, then compute the derivative directly, but you’d have to decide whether \((1, 1)\) is on the part of the graph represented by the positive square root or the part represented by the negative square root. Implicit differentiation is an easier way to find the slope of the tangent line. Remember to use the chain rule when differentiating the left-hand side.

\[
 y^2 = 2x^3 - x^4
\]

\[
 2yy' = 6x^2 - 4x^3
\]

\[
 y' = \frac{6x^2 - 4x^3}{2y} = \frac{3x^2 - 2x^3}{y}
\]

This is the formula for the slope of the tangent line to the curve at the point \((x, y)\). Now put in \( x = 1 \) and \( y = 1 \) to see that the slope of the tangent line to the curve at \((1, 1)\) is 1. Since the tangent line at \((1, 1)\) certainly must pass through \((1, 1)\) itself, an equation for the tangent line is \( y - 1 = x - 1 \) or \( y = x \).

(b) At what points does this curve have horizontal tangent lines? [2 points]

**Solution:** The tangent line is horizontal if it has slope 0, so we set \( y' = 0 \) in the equation derived in part (a).

\[
 0 = \frac{3x^2 - 2x^3}{y}
\]

\[
 0 = \frac{3x^2 - 2x^3}{y}
\]

\[
 0 = x^2(3 - 2x)
\]

So \( x = 0 \) or \( x = \frac{3}{2} \). Putting these values of \( x \) back into the formula for the curve, we find the corresponding values for \( y \). When \( x = \frac{3}{2} \), \( y^2 = 2 \cdot (\frac{3}{2})^3 - (\frac{3}{2})^4 = \frac{27}{16} \), so \( y = \frac{3\sqrt{3}}{4} \). So our calculation above with \( y' \) said that at the point \((3/2, 27/16)\), we had \( y' = \frac{0}{27/16} = 0 \). However, when \( x = 0 \), \( y = 0 \) too. At that point, our calculation with \( y' \) says \( y' = \frac{0}{0} \), which is undefined. So the only actual horizontal tangent is at \((3/2, 27/16)\).
(c) At what point does this curve have a vertical tangent line? [2 points]

**Solution:** The curve has a vertical tangent line whenever the slope represented by $y'$ goes to $\infty$, so whenever $y = 0$. Looking back at the original equation, $y = 0$ means that $x^3(x - 2) = 0$, so $x = 0$ or $x = 2$. As we saw above, if $x = 0$, then $y' = \frac{0}{x}$ and the slope is undefined, rather than $\infty$. So, the only vertical tangent line is at $(2, 0)$.

(d) At what point is this curve not differentiable? [4 points]

**Solution:** The curve is definitely differentiable as long as the denominator of the formula above for $y'$ is not 0, so we just have to figure out what happens if $y = 0$. Using the original formula for the curve, if $y = 0$, then $x$ must satisfy

$$0 = x^3(2 - x),$$

so $x = 2$ or $x = 0$. If $x = 2$, then the formula for $y'$ looks like $\frac{-8}{16}$, which means the tangent line is vertical. If $x = 0$, then the formula looks like $\frac{0}{0}$, which means the slope of the tangent line is undefined. On the graph, this means that there is a cusp at $(0,0)$. So, the curve is definitely not differentiable at $(0,0)$. When grading this problem, full credit was given for the answer $(0,0)$ or for the answer $(0,0)$ and $(2,0)$, since vertical tangent lines are sometimes defined to be points of non-differentiability.

(e) Which of the graphs in Figure 1 could be the graph of this curve? Justify your answer. [5 points]

**Solution:** The middle graph is correct. The correct graph should have a cusp at $(0,0)$, which rules out the top graph. It should also have a vertical tangent line at $x = 2$, which the bottom graph does not. That leaves the middle graph.

![Figure 1: Graphs for Question 5(e)](image)
6. (a) Let \( g(x) = \int_{3}^{x} t^4 \, dt \). Find \( g'(x) \). (Be sure to justify your answer.) [6 points]

**Solution:** Apply the fundamental theorem of calculus, part 1, which reads as follows.

**Theorem.** Let \( f \) be continuous on \([a, b]\). Then the function

\[
g(x) = \int_{a}^{x} f(t) \, dt
\]

is well-defined and differentiable on \([a, b]\) and \( g'(x) = f(x) \).

Therefore, \( g'(x) = x^x \).

(b) Find \( g''(x) \). [12 points]

**Solution:** Using the solution from part (a), we need to find the derivative of \( g'(x) = x^x \). Use logarithmic differentiation to do so. Set up the derivative like this:

\[
y = x^x \\
\ln y = \ln (x^x) \\
\ln y = x \ln x
\]

Now differentiate, using the chain rule on the left and the product rule on the right.

\[
\frac{y'}{y} = \ln x + \frac{x}{x} \\
y' = y (\ln x + 1) \\
y' = x^x (\ln x + 1)
\]

7. An spheroid is a solid with elliptical cross-sections in two coordinate directions and circular cross-sections in the third direction. In other words, it’s a sphere that has been stretched along one axis. Find the volume of the spheroid shown below, where the elliptical cross-sections have minor axis \( a \) and major axis \( b \) and the circular cross-sections have radius \( a \). [Hint: the formula \( \frac{x^2}{b^2} + \frac{y^2}{a^2} = 1 \) might be useful.] [25 points]
**Solution:** Situate the spheroid in the coordinate axes as shown above. Then, slicing vertically produces disks as cross-sections and slicing horizontally produces ellipses as cross-sections. We could also use cylindrical shells, slicing horizontally, with the \( x \)-axis as the innermost shell.

**Disks:** There is a disk cross-section for each \( x \) between \(-b\) and \( b\), but we can just count those between 0 and \( b\) and multiply the whole expression by 2. So the appropriate integral here is \( V = 2 \int_0^b A(x) \, dx \), where \( A(x) \) is the area of the disk at cross-section \( x \). The radius of the disk at cross-section \( x \) is given by the \( y \)-value in the formula for the ellipse \( \frac{x^2}{b^2} + \frac{y^2}{a^2} = 1 \). So it’s \( y = \pm \sqrt{a^2 - \frac{a^2 x^2}{b^2}} \). Therefore, the area of the disk at cross-section \( x \) is \( A(x) = \pi \left( a^2 - \frac{a^2 x^2}{b^2} \right) \). Putting everything together:

\[
V = 2 \int_0^b \pi \left( a^2 - \frac{a^2 x^2}{b^2} \right) \, dx \\
= 2 \pi a^2 \int_0^b \left( 1 - \frac{x^2}{b^2} \right) \, dx \\
= 2 \pi a^2 \left( x - \frac{x^3}{3b^2} \right) \bigg|_0^b \\
= 2 \pi a^2 \left( b - \frac{b^3}{3b^2} \right) \\
= \frac{4\pi}{3} a^2 b.
\]

**Cylindrical Shells:** The spheroid can also be sliced into cylindrical shells, as described above. There is one shell for each \( y \)-value between 0 and \( a\), so the integral is \( dy \) and goes from 0 to \( a\). That is, \( V = \int_0^a A(y) \, dy \), where \( A(y) \) is the area of the shell at distance \( y \) from the central shell (which is the \( x \)-axis). The radius of such a shell is given by \( y \) and its height comes from the formula for the ellipse in the \( xy \)-plane: \( \frac{x^2}{b^2} + \frac{y^2}{a^2} = 1 \). Solving for \( x \) in this formula gives \( x = \pm \sqrt{b^2 - \frac{b^2 y^2}{a^2}} \), but the shell extends equally far to the right and the left of the \( y \)-axis, so the height of the cylinder is actually \( 2 \sqrt{b^2 - \frac{b^2 y^2}{a^2}} \). Therefore, \( A(y) = 2\pi y \cdot 2 \sqrt{b^2 - \frac{b^2 y^2}{a^2}} \). Putting everything together:

\[
V = \int_0^a 4\pi y \sqrt{b^2 - \frac{b^2 y^2}{a^2}} \, dy \\
= 4\pi b \int_0^a y \sqrt{1 - \frac{y^2}{a^2}} \, dy \\
\]

To evaluate this integral, use the substitution \( u = 1 - \frac{y^2}{a^2} \). Then \( du = -\frac{2y}{a^2} \), so the \( u \)-version of the integral is as follows (switching to indefinite integrals temporarily).

\[
= -2\pi a^2 b \int \sqrt{u} \, du \\
= -\frac{4}{3} \pi a^2 b u^{3/2} \\
= 7
\]
Converting back from $u$ to $y$ and substituting the original limits of integration, we have

$$\left[-\frac{4}{3} \pi a^2 b \left(1 - \frac{y^2}{a^2}\right)\right]^a_0 = -\frac{4}{3} \pi a^2 b \left(1 - \frac{a^2}{a^2}\right) + \frac{4}{3} \pi a^2 b = \frac{4}{3} \pi a^2 b.$$ 

8. The graph in Figure 2 will be used in this problem and the next one. Suppose that it is the graph of a function $f$.

(a) At approximately what values of $x$ does the antiderivative of $f$ have local maxima or minima? Specify which values correspond to maxima and which correspond to minima. [3 points]

**Solution:** Let $F$ be the antiderivative of $f$, so $F' = f$. Then $F$ has a local max or min whenever $f$ is 0. According to the graph, that’s around $x = -2.5$ and $x = 6.5$. Also, $F$ has a local max whenever it is switching from increasing to decreasing, so whenever $f$ is switching from positive to negative. According to the graph, that’s at $x = -2.5$. Likewise, $F$ has a local min if it is switching from decreasing to increasing, so if $f$ is switching from negative to positive. According to the graph, that’s at $x = 6.5$.

(b) At approximately what values of $x$ does the antiderivative of $f$ have inflection points? [3 points]

**Solution:** Inflection points occur when a function’s second derivative is 0 and changes from positive to negative or vice versa at that point. So, $F$ has an inflection point at $x$ if $f''(x) = 0$ and changes from positive to negative or vice versa at $x$. If $f''(x) = 0$, that means that the slope of the tangent line to the graph of $f$ at $x$ is 0. The only place where $f$ has a horizontal tangent line is $x = 2$. When $x$ is a little less than 2, the slope of the tangent line is negative. When $x$ is a little more than 2, the slope of the tangent line is positive. So $f''(2) = 0$ and $f''$ changes sign at $x = 2$. Therefore, $F$ has an inflection point there.

(c) Using the graph paper in Figure 3, sketch a graph of an antiderivative of $f$ that clearly shows the following features:

- where it is increasing or decreasing [4 points]
- local maxima and minima [4 points]
- inflection points and concavity [4 points]

(Hint: The antiderivative of $f$ will have vertical asymptotes in the same places that $f$ does.)

**Solution:** See Figure 3 for the correct graph. The graph of $F$ should be increasing on $(-\infty, -2.5)$ and $(-2, 6)$ and $(6.5, \infty)$. It should be decreasing on $(-2.5, -2)$ and $(6, 6.5)$. It should have a local maximum at $x = -2.5$ and a local minimum at $x = 6.5$. It should have only one inflection point, at $x = 2$, where it changes from concave upward to concave downward.

[P.S. The hint in this problem was not quite correct, although it was impossible to notice just based on the graph provided on the exam. The vertical asymptotes in the graph of $f$ (Figure 2) indicate that the graph of its antiderivative, $F$, will have slope approaching $\infty$ as $x \to -2$ or as $x \to 6$ from one side and $-\infty$ as $x \to -2$ or as $x \to 6$ from the other side. This behavior could occur if $F$ had vertical asymptotes at $x = -2$ and $x = 6$, or it could occur if $F$ had cusps at those points, as shown in Figure 3. Based on the actual function I used to make the graph of $f$, the cusps are correct. However, it’s impossible to know this without a formula for $f$. Of course, full credit was given for answers with vertical asymptotes!]
Figure 2: Graph for Questions 8 and 9

Figure 3: Graph of the antiderivative of the function in Figure 2
9. Refer again to the graph of \( f \) in Figure 2. Which of the following functions could be the function shown in that graph? Justify your answer using evidence from the graph. [8 points]

(a) \( f(x) = \sin x + \cos x \)
   The graph shown is not periodic, so this is not a good choice.

(b) \( f(x) = 6 - \frac{1}{x+2} - \frac{1}{x+6} \)
   The graph of this function would have vertical asymptotes \( x = 2 \) and \( x = 6 \) whereas the actual graph has vertical asymptotes at \( x = -2 \) and \( x = 6 \), so this can’t be the right choice.

(c) \( f(x) = \frac{x^2 + 2x - 3}{(x+2)(x-6)} \)
   The graph of this function would have vertical asymptotes in the right places, but would tend to \( \infty \) and \( -\infty \), respectively, as \( x \to \infty \) and \( x \to -\infty \). The graph in the picture tends to 2.

(d) \( f(x) = 2 + \frac{1}{x+2} - \frac{1}{x-6} \)
   The graph of this function would have vertical asymptotes in the right places, but the \( e^x \) term means that its limit as \( x \to \infty \) is definitely \( \infty \), whereas the graph shown tends to 2 as \( x \to \infty \).

(e) \( f(x) = 2 + \frac{1}{x^2} - \frac{1}{x-6} \)
   This is the correct choice. It is the only function in the list with the correct vertical and horizontal asymptotes. There are other ways to deduce that this is the correct answer. Full credit was given for any explanation that identified the correct graph uniquely.

10. Let \( f(x) = \frac{1}{x} \) for \( x > 0 \) and \( f(0) = 0 \).

   (a) Write a formula for the Riemann sum with 4 rectangles for \( f \) on the interval \([0,1]\) using the right endpoints of the sub-intervals. Draw a graph that illustrates the rectangles you used. [6 points]

   **Solution:** To receive full credit, your graph needed to show a curve that looked like \( 1/x \), show the interval \([0,1]\) divided evenly into four rectangles, and show rectangles constructed using right endpoints. The Riemann sum is

   \[
   R_4 = \frac{1}{4} \cdot f\left(\frac{1}{4}\right) + \frac{1}{4} \cdot f\left(\frac{1}{2}\right) + \frac{1}{4} \cdot f\left(\frac{3}{4}\right) + \frac{1}{4} \cdot f(1)
   \]

   \[
   = \frac{1}{4} \cdot 4 + \frac{1}{4} \cdot 2 + \frac{1}{4} \cdot \frac{4}{3} + \frac{1}{4} \cdot 1
   \]

   \[
   = \frac{25}{12}
   \]
(b) Use sigma (Σ) notation to write a formula for the Riemann sum with \( n \) rectangles of \( f \) on the interval \([0, 1]\) using the right endpoints of the sub-intervals. Before you simplify your formula, show which part of it corresponds to height of a rectangle and which part corresponds to width of a rectangle. Then simplify as much as possible. [10 points]

**Solution:** The formula follows the pattern from part (a). The height of a rectangle is given by \( f(x_i) \) and the width is \( \frac{1}{n} \). The Riemann sum simplifies as follows.

\[
R_n = \sum_{i=1}^{n} f(x_i) \cdot \frac{1}{n} \\
= \sum_{i=1}^{n} f \left( \frac{i}{n} \right) \cdot \frac{1}{n} \\
= \sum_{i=1}^{n} \frac{n}{i} \cdot \frac{1}{n} \\
= \sum_{i=1}^{n} \frac{1}{i}
\]

(c) Use your answer in (b) to explain why \( f \) is not integrable on the interval \([0, 1]\). [8 points]

**Solution:** This part of the problem was not graded, which is why the final exam grade was out of 192 points instead of 200. The correct answer to (b) does not lend itself to a good explanation of why \( f \) is not integrable on \([0, 1]\). My apologies for any confusion!

11. A rectangular poster is to be produced with a design in the middle and blank space around the edges. The blank space will extend 4 cm from the top and bottom edges and 2 cm from the left and right edges. The area of the design in the middle will be 100 cm\(^2\). Find the dimensions of the poster with the smallest total area. [20 points]

**Solution:** Let \( x \) and \( y \) represent the dimensions of the design in the middle of the poster. The area of the design is represented by \( xy \), so we know that \( xy = 100 \). The area of the total poster is given by \((x+4)(y+8)\). Solve \( xy = 100 \) for \( y \) and substitute into the formula for the area of the total poster to get a function:

\[
A(x) = (x + 4) \left( \frac{100}{x} + 8 \right) = 100 + 8x + \frac{400}{x} + 32
\]

This is the function that we need to minimize. So, take its derivative with respect to \( x \), set to zero, and solve.

\[
A'(x) = 8 - \frac{400}{x^2}
\]

\[
A'(x) = 0 \text{ if } 8 = \frac{400}{x^2} \text{ so } x^2 = 50 \text{ so } x = \sqrt{50} = 5\sqrt{2}
\]

To double check that this is a minimum, not a maximum, you could check that

\[
A''(x) = \frac{800}{x^3},
\]

which is positive when \( x = 5\sqrt{2} \). The final answer is that the poster with smallest total area has \( x = 5\sqrt{2} \) and \( y = \frac{100}{5\sqrt{2}} = 10\sqrt{2} \), which corresponds to total dimensions of \( 5\sqrt{2} + 4 \) cm by \( 10\sqrt{2} + 8 \) cm.
Extra Credit (10 points each question)

Solutions? Sorry, I can't give away all my secrets...but I'll talk about these problems with anyone who is interested.

EC1) A fixed point of a function $f$ is a number $c$ in the domain of $f$ such that $f(c) = c$. (That is, the function $f$ doesn’t move $c$; it leaves it fixed.)

(a) Sketch the graph of a continuous function with domain $[0, 1]$ and range contained in $[0, 1]$. Show where the function you drew has a fixed point. [2 points]

(b) Use the intermediate value theorem to prove that any continuous function with domain $[0, 1]$ and range contained in $[0, 1]$ must have a fixed point. [Hint: Try to draw a function with domain and range $[0, 1]$ that does not have a fixed point. What's the obstacle to doing so?] [8 points]

EC2) In this problem, you will prove that the area of a circle with radius $r$ is $\pi r^2$.

(a) Calculate the area of a regular polygon (regular means all sides have the same length) with $n$ sides inscribed in a circle of radius $r$. (Hint: split the polygon into triangles.) [2 points]

(b) Let $n$ go to $\infty$ in the formula you found in part (a). [5 points]

(c) Explain why taking the limit in part (b) calculates the area of a circle. [3 points]