COLUMBIA UNIVERSITY
Intro to Numerical Methods
APAM E4300 (1)
FINAL EXAM – MAY 13, 2013
INSTRUCTOR: DR. SANDRO FUSCO

FAMILY NAME: ______________________________________________________________
GIVEN NAME:  ______________________________________________________________

INSTRUCTIONS:

1. You have 2 hours 50 minutes.
2. Attempt all seven (7) questions. Then try the Extra Credit Question.
3. Your work must justify the answer you give.
4. Point values are as shown. Work is required for full credit, and may earn partial credit.
5. Be sure to indicate your answer VERY clearly. Put your answer in a circle or box when applicable.
6. No lecture notes and/or books are permitted. Calculators are allowed.
7. This is the first of nine (9) pages.

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Problem 1: (10 Points)

a) [2 points] For a general root finding problem, list the following three algorithms in order of increasing speed (where by faster we mean takes less steps to converge to an answer):

   Secant method, Newton’s method, Bisection method

b) [2 points] Write the equation for the tangent line to \( y = f(x) \) at \( x = p \).

c) [2 points] Solve for the x-intercept of the line in point (b). What formula have you derived, with what roles for \( p \) and \( x \)?

d) [2 points] Write the equation of the line that intersects the curve \( y = f(x) \) at \( x = p \) and \( x = q \).

e) [2 points] Solve for the x-intercept of the line in point (d). What formula have you derived, with what roles for \( p \), \( q \), and \( x \)?
Problem 2: (15 Points)
Let \( n \) and \( m \) be positive integers and consider the binary floating point system where our machine stores numbers of the following form:

\[ \pm 1.a_1 a_2 \ldots a_n 2^e \] where \( e \in \{0, -1, -2, \ldots, -m\} \) and \( a_k \in \{0, 1\} \) for all \( k \).

Remember: In a binary system each \( a_k \) gets multiplied by \( 2^{-k} \) and all these get added up to produce the number being represented.

a) [5 points] In terms of \( n \) and \( m \), how many numbers are in this system? Don’t forget the ±

b) [5 points] Let \( m=2 \) and \( n=5 \). What is machine precision if we approximate all numbers by rounding to the closest number in the floating point system?

c) [5 points] Let \( m=2 \). Find the smallest value of \( n \) that will ensure that some number in the system is bigger than 1.8.
Problem 3: (15 Points)

Write the following matrix in the form $LU$, where $L$ is a unit lower triangular matrix and $U$ is an upper triangular matrix:

$$
\begin{pmatrix}
4 & -1 & -1 \\
-1 & 4 & -1 \\
-1 & -1 & 4
\end{pmatrix}
$$

Write the same matrix in the form $LL^T$, where $L$ is lower triangular.
Problem 4: (15 Points)

Determine all the values of $a$, $b$, $c$, $d$, $e$, and $f$ for which the following function is a cubic spline:

$$ s(x) = \begin{cases} 
    ax^2 + b(x - 1)^3 & x \in (-\infty, 1] \\
    cx^2 + d & x \in [1, 2] \\
    ex^2 + f(x - 2)^3 & x \in [2, \infty) 
\end{cases} $$
Problem 5: (15 Points)

a) [10 points] In class we have seen one way to approximate the derivative of a function $f$:

$$f'(x) \approx \frac{f(x + h) - f(x)}{h}$$

for some small number $h$ (forward finite difference). Assuming that $f \in C^2$, use Taylor’s Theorem to determine the accuracy of this approximation.

b) [5 points] Show that, with this formula, we can approximate a derivative to only about the square root of the machine precision.
Problem 6: (15 Points)

Consider the integration formula,

$$\int_{-1}^{1} f(x) \, dx \approx f(\alpha) + f(-\alpha).$$

(a) For what value(s) of $\alpha$, if any, will this formula be exact for all polynomials of degree 1 or less?

(b) For what value(s) of $\alpha$, if any, will this formula be exact for all polynomials of degree 3 or less?

(c) For what value(s) of $\alpha$, if any, will this formula be exact for all polynomials of the form $a + bx + cx^3 + dx^4$, where $a$, $b$, $c$, and $d$ are constants?
Problem 7: (15 Points)

Find a formula of the form

\[ \int_0^1 xf(x) \, dx \approx A_0 f(x_0) + A_1 f(x_1), \]

that is exact for all polynomials of degree 3 or less.
Problem EC1: (10 Extra Points)

Write down the result of applying one step of Euler’s method to the initial value problem $y' = (t + 1)e^{-y}$, $y(0) = 0$, using step size $h = 0.1$. Do the same for the midpoint method and for Heun’s method.

Enjoy the summer break!