SUMMARY
The purpose of this document is to provide a possible foundation that the projected forward LIBOR is independent of the collateral currency, provided that the “spread of the spread” is uncorrelated to the forward LIBOR.

Result: The projected forward of the currency $i$ benchmark interest rate index LIBOR $L^{(i)}(T)$ under collateral currency $k$ is independent of currency $k$ if and only if the variables $e^{\int_0^T -y^{(k)}(s) \, ds}$ and $L^{(i)}(T)$ are uncorrelated. In this case, the projected forward LIBOR is given by the usual expression $E^T \left[ L^{(i)}(T) \right]$.

Details
In the market there is a convention that the projected forward LIBOR is independent of the collateral currency. This document attempts to provide a theoretical justification of it and lay out the necessary and sufficient condition.

In [MFAT]: Derivative pricing formula when the payoff is in currency $i$ while collateral is in currency $k$ is given by

$$h^{(i)}(t) = E^Q \left[ e^{-\int_0^T r^{(i)}(s) \, ds} e^{\int_0^T r^{(i)}(s) \, ds} h^{(i)}(T) \right]$$

where $r^{(i)}(t)$ is the risk-free continuous compounding zero rate for currency $i$ at time $t$, $c^{(k)}(t)$ be the continuous compounding collateral return rate for at time $t$. and $y^{(k)}(t) = r^{(k)}(t) - c^{(k)}(t)$, $h^{(i)}$ is the payoff.

That is $h^{(i)}(t) = D^{(i)}(t,T) E^T \left[ e^{\int_0^T -y^{(i)}(s) \, ds} h^{(i)}(T) \right]$ (see equation 5 of [MFAT])

where $D^{(i)}(t,T) = E^Q \left[ e^{-\int_t^T r^{(i)}(s) \, ds} \right]$ and $y^{(i,k)}(t) = y^{(i)}(t) - y^{(k)}(t)$

Consider we have a discount curve that is bootstrapped to discount cash flows of currency $i$ when collateral is in currency $k$. We can define $e^{-R^{(i,k)}(t)(T-t)} = DF(t,T) = D^{(i)}(t,T) E^T \left[ e^{\int_0^T -y^{(i,k)}(s) \, ds} \right]$ where $R^{(i,k)}$ is the resulting discount zero curve that we already bootstrapped.
Consider now a payoff of the currency $i$ benchmark interest rate index LIBOR $L^{(i)}(T)$ at time $T$ in currency $i$ when collateral is in currency $k$.

According to the formula, we will have $h^{(i)}_t = D^{(i)}(t,T)E^T \left[ e^{\int_t^T -y^{(i,k)}(s)ds} L^{(i)}(T) \right]$. From the way that LIBOR projected forward is used for pricing, this expression should be the same as $e^{-r^{(i,i)}(T-T-\tau)} PL^{(i)}(T)$ where $PL$ stands for projected LIBOR. Equating the two, we get

$$PL^{(i)}(T) = e^{r^{(i,i)}(T-T-\tau)} D^{(i)}(t,T)E^T \left[ e^{\int_t^T -y^{(i,i)}(s)ds} L^{(i)}(T) \right]$$

$$D^{(i)}(t,T) E^T \left[ e^{\int_t^T -y^{(i,i)}(s)ds} L^{(i)}(T) \right] = D^{(i)}(t,T) E^T \left[ e^{\int_t^T -y^{(i,k)}(s)ds} \right]$$

$$E^T \left[ e^{\int_t^T -y^{(i,i)}(s)ds} L^{(i)}(T) \right] = E^T \left[ e^{\int_t^T -y^{(i,k)}(s)ds} \right]$$

This means, the necessary and sufficient condition for $PL^{(i)}(T) = E^T \left[ L^{(i)}(T) \right]$, the projected LIBOR of the above expression while $k = i$, is to have

$$E^T \left[ e^{\int_t^T -y^{(i,i)}(s)ds} \right] E^T \left[ L^{(i)}(T) \right] = E^T \left[ e^{\int_t^T -y^{(i,k)}(s)ds} L^{(i)}(T) \right].$$

In other words, the variables $e^{\int_t^T -y^{(i,i)}(s)ds}$ and $L^{(i)}(T)$ are uncorrelated (in the $T^i$ forward measure).

This seems plausible since $y^{(i,k)}(t) = y^{(i)}(t) - y^{(k)}(t) = r^{(i)}(t) - c^{(i)}(t) - \left( r^{(k)}(t) - c^{(k)}(t) \right)$ is a spread of a spread, which may intuitively have nothing to do with currency $i$ LIBOR.

Reference


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