Here are some problems on the Seiberg-Witten equations related to the material of Lectures 3 and 4.

**Problem 1.** Given a spin$^c$ structure on a 4-manifold, show that real-valued self-dual forms act via the Clifford multiplication as elements of $\mathfrak{su}(S^+) \subset \text{End}(S)$.

**Problem 2.** Show that if $(A, \Phi)$ is a solution to the Seiberg-Witten equations, then for every gauge transformation $u : X \to S^1$, the configuration $u \cdot (A, \Phi)$ is a solution too.

**Problem 3.** Show that on an oriented Riemannian 4-manifold the operator

$$d^* + d^+ : \Omega^1 \to \Omega^0 \oplus \Omega^+$$

is elliptic. Use the Hodge theorem to show that it has index $b_1 - (1 + b^+)$.

**Problem 4.** Consider the torus $T^4$ equipped with a flat metric.

- Show that there exists a spin$^c$ structure $\mathfrak{s}$ on $T^4$ for which the spinor bundles $S^\pm$ are trivial.
- Describe the space of self-dual harmonic forms
- Exhibit an explicit self-dual form $\omega^+$ for which the perturbed Seiberg-Witten equations do not have reducible solutions.

**Problem 5.** Use Fourier series as in Problem Session 1 to show that the natural inclusion of $L^2_1(S^1)$ into $L^2(S^1)$ is compact (i.e. the image of bounded sets is precompact). Hint: try to construct a convergent subsequence by hand using a diagonal argument.

**Bonus problem 1.** In the case in which $b_1(X)$ is not necessarily zero, show that each homotopy class of maps $X \to S^1$ contains exactly one $S^1$-family of gauge transformations $u$ for which $u \cdot A$ is in Coulomb gauge with respect to the base connection $A_0$. Using the correspondence between such homotopy classes and elements of $H^1(X; \mathbb{Z})$, show that in general $\mathcal{B}^*(X, \mathfrak{s})$ is homotopy equivalent to $T^{b_1(X)} \times \mathbb{C}P^\infty$. 

1
**Bonus problem 2.** On an oriented Riemannian 4-manifold $M$, consider the operator $\varepsilon$ on $\Omega^*(M)$ acting on $\Omega^p$ as $(-1)^{\frac{p(p-1)}{2}+1}\ast$.

- Show that $\varepsilon^2$ is the identity; denote the $\pm 1$ eigenspaces by $A_+$ and $A_-$. 
- Show that $d + d^*$ anticommutes with $\varepsilon$, so that it defines an operator $A_+ \rightarrow A_-$. 

Use the Hodge theorem to show that the operator $d + d^* : A_+ \rightarrow A_-$ has index $\sigma(M)$. Hint: try to match odd degree harmonic forms in $A_+$ and $A_-$ using $\varepsilon$.

**Bonus bonus problem.** Consider a compact oriented 4-manifold $X$ equipped with a scalar flat metric, i.e. $s \equiv 0$, and assume that $X$ admits a spin$^c$ structure $(S, \rho)$ with $c_1(S^+) \equiv 0$ torsion.

- Show that there is a spin$^c$ connection $A_0$ with $A_0^0$ flat. 
- Show that the kernel of $D_{A_0}^+$ consists of $A_0$-parallel sections. 
- Conclude that $\sigma(X) \geq -16$. Is the inequality sharp?