AN INTRODUCTION TO SEIBERG-WITTEN INVARIANTS - PROBLEM SESSION 3

Here are some problems on the Seiberg-Witten equations related to the material of Lectures 3 and 4.

Problem 1.* Given a spin^c structure on a 4-manifold, show that real-valued self-dual forms act via the Clifford multiplication as elements of $\mathfrak{su}(S^+) \subset \mathrm{End}(S)$.

Problem 2.* Show that if (A, Φ) is a solution to the Seiberg-Witten equations, then for every gauge transformation $u : X \to S^1$, the configuration $u \cdot (A, \Phi)$ is a solution too.

Problem 3. Show that on an oriented Riemannian 4-manifold the operator

$$d^* + d^+ : \Omega^1 \to \Omega^0 \oplus \Omega^+$$

is elliptic. Use the Hodge theorem to show that it has index $b_1 - (1 + b^+)$.

Problem 4.* Consider the torus T^4 equipped with a flat metric.

- Show that there exists a spin^c structure $\mathfrak{s}$ on T^4 for which the spinor bundles $S^\pm$ are trivial.
- Describe the space of self-dual harmonic forms
- Exhibit an explicit self-dual form ω^+ for which the perturbed Seiberg-Witten equations do not have reducible solutions.

Problem 5. Use Fourier series as in Problem Session 1 to show that the natural inclusion of $L^2_1(S^1)$ into $L^2(S^1)$ is compact (i.e. the image of bounded sets is precompact). Hint: try to construct a convergent subsequence by hand using a diagonal argument.

Bonus problem 1. In the case in which $b_1(X)$ is not necessarily zero, show that each homotopy class of maps $X \to S^1$ contains exactly one S^1-family of gauge transformations u for which $u \cdot A$ is in Coulomb gauge with respect to the base connection A_0. Using the correspondence between such homotopy classes and elements of $H^1(X; \mathbb{Z})$, show that in general $\mathcal{B}^*(X, \mathfrak{s})$ is homotopy equivalent to $T^{b_1(X)} \times \mathbb{C}P^\infty$.

Bonus problem 2. On an oriented Riemannian 4-manifold M, consider the the operator ε on $\Omega^*(M)$ acting on Ω^p as $(-1)^{\frac{p(p-1)}{2}+1}*$.

- Show that ε^2 is the identity; denote the ± 1 eigenspaces by A_+ and A_-.
- Show that $d+d^*$ anticommutes with ε, so that it defines an operator $A_+ \to A_-$.

Use the Hodge theorem to show that the operator $d + d^* : A_+ \to A_-$ has index $\sigma(M)$. Hint: try to match odd degree harmonic forms in A_+ and A_- using ε.

Bonus bonus problem. Consider a compact oriented 4-manifold X equipped with a scalar flat metric, i.e. $s \equiv 0$, and assume that X admits a spin^c structure (S, ρ) with $c_1(S^+)$ torsion.

- Show that there is a spin^c connection A_0 with A_0^t flat.
- Show that the kernel of $D^+_{A_0}$ consists of A_0-parallel sections.
- Conclude that $\sigma(X) \geq -16$. Is the inequality sharp?