AN INTRODUCTION TO SEIBERG-WITTEN INVARIANTS - PROBLEM SESSION 2

Here are some problems on Clifford algebras, the associated bundles, and Dirac operators. You are encouraged to work on the starred problems first, as they will be particularly relevant for the minicourse. In case you finish early, you can find some fun bonus questions at the end.

Problem 1*. Using Pauli matrices, show that the isomorphism of $\mathbb{C}$-algebras

$$\mathbb{H} \otimes_{\mathbb{R}} \mathbb{C} \cong M(2, \mathbb{C})$$

holds. This shows that $\mathrm{Cl}(\mathbb{R}^2) \otimes \mathbb{C} \cong M(2, \mathbb{C})$. How do you interpret the $\mathbb{Z}/2$-grading under this identification?

Problem 2*. In this exercise we study the Clifford algebra $\mathrm{Cl}(\mathbb{R}^4)$ and its spinor representation.

- For an oriented orthonormal basis $u_1, \dots, u_4$ of $\mathbb{R}^4$. Show that the element $c = u_1 \cdot u_2 \cdot u_3 \cdot u_4 \in \mathrm{Cl}(R^4)$ is independent of the choice of oriented orthonormal basis, and corresponds to the volume form under the identification (as vector spaces) with $\Lambda^* \mathbb{R}^4$.
- Compute the action of c on the spinor representation.

Using the previous points, provide an intrinsic characterization of the splitting $S = S^+ \oplus S^-$ of the spinor bundle of a spin^c structure on a 4-manifold. Conclude that the Dirac operator D splits in the two chiral parts $D^\pm : S^\pm \to S^\mp$.

Problem 3*. For a Clifford bundle S with connection ∇^S, show that the principal symbol of the associated Dirac operator D is given by the Clifford multiplication

$$P_{D,x}(\xi)s = \xi \cdot s.$$

Conclude that D is elliptic by looking at the square of the symbol.

Problem 4. Recall that for a complex line bundle L on a closed surface Σ, $c_1(L)$ is the signed count of zeroes of a generic section. Consider a spin^c structure $S \to \Sigma$, which we know naturally splits in a direct sum of line bundles $S = S^+ \oplus S^-$. Show that $c_1(S^+) - c_1(S^-) = 2g(\Sigma) - 2$ by taking a generic vector field X on Σ and looking at its action $S^+ \to S^-$ via Clifford multiplication.

Problem 5. In this exercise you will show that a Kähler 2-manifold admits a canonical spin^c structure. Consider a Riemann surface Σ equipped with a Kähler metric. Show that $S = \Omega^0 \oplus \Omega^{0,1}$ is naturally a Clifford bundle, where the action of $v \in T^*\Sigma$ is given by

$$v \cdot (f, \alpha) = \sqrt{2}(-\langle \alpha, v^{0,1} \rangle, f v^{0,1}).$$

Here $v^{0,1}$ is the projection of v under $T^*\Sigma \subset T^*\Sigma \otimes \mathbb{C} \to T^{0,1}X$. Why is there a factor of $\sqrt{2}$?

Bonus question 1. Show that $\mathrm{Cl}(\mathbb{R}^3)$ is not a division algebra, i.e. there are elements $x, y \neq 0$ such that $xy = 0$.

Bonus question 2. In Problem 5, show that the corresponding Dirac operator is a multiple of $\bar{\partial} + \bar{\partial}^*$. What is the index of D^+?