AN INTRODUCTION TO SEIBERG-WITTEN INVARIANTS - PROBLEM SESSION 1

Here are some problems about Sobolev spaces, elliptic operators and the Hodge theorem. You are encouraged to work on the starred problems, as they will be particularly relevant for the minicourse. In case you finish early, you can find some fun bonus questions at the end.

Problem 1*. We can study the properties of the Sobolev space $L^2_1(S^1)$ using the Fourier series expansion of $f = \sum_{n\in\mathbb{Z}} a_n e^{inx}$ where

$$a_n = \frac{1}{2\pi}\int_0^{2\pi} f(x)e^{-inx}dx.$$

Recall the Parseval identity $\frac{1}{2\pi}\|f\|^2_{L^2} = \sum_{n\in\mathbb{Z}} |a_n|^2$.

- Show that for a smooth f the Fourier coefficients of f' are given by $\{in\cdot a_n\}$, and therefore the L^2_1 Sobolev norm satisfies

$$\frac{1}{2\pi}\|f\|^2_{L^2_1} = \sum_{n\in\mathbb{Z}}(1+n^2)|a_n|^2.$$

- Use this to show that there exists a constant $C > 0$ (independent of the smooth function f) for which $|f(x)| \leq C\|f\|_{L^2_1}$ for all x. Can you conclude that $L^2_1(S^1)$ embeds continuously in $C^0(S^1)$?

What happens in the case of the n-torus?

Problem 2. If $B_{1/2}$ denotes the ball of radius $1/2$ in $\mathbb{R}^2$, not all functions in $L^2_1(B_{1/2})$ are continuous! To show this, show that the radial function $\log\log(1/r)$ defined on $B_{1/2}\setminus\{0\}$ is in $L^2_1(B_{1/2})$.

Problem 3*. Let X, Y, Z be Hilbert spaces, and consider operators $T : X \to Y$ and $K : Y \to Z$ respectively bounded and compact such that the inequality

$$\|x\|_X \leq C(\|Tx\|_Y + \|Kx\|_Z)$$

holds for some $C > 0$. Show that $\mathrm{Ker}(T)$ is finite dimensional; use the fact that the unit sphere of a Hilbert space V is compact if and only if V is finite dimensional. Conclude that the kernel of an elliptic operator on a compact manifold is finite dimensional.

Problem 4*. Show that d^* is the adjoint of d with respect to the L^2-inner product, i.e. for $\gamma \in \Omega^{p-1}$ and $\alpha \in \Omega^p$,

$$\langle d\gamma, \alpha\rangle_{L^2} = \langle \gamma, d^*\alpha\rangle_{L^2}.$$

To do this, apply Stokes' theorem to $\gamma \wedge *\alpha \in \Omega^{n-1}$.

Problem 5. Use the intrinsic definition to show that the principal symbol of $d + d^*$ is

$$P_{d+d^*,x}(\xi)\alpha = \xi \wedge \alpha + \xi \lrcorner \alpha,$$

where the convention for $\lrcorner$ is the one given in class. Conclude that $d+d^* : \Omega^* \to \Omega^*$ is elliptic.

Bonus question 1. Show that the elliptic operator $\frac{d}{dt} : L^2_1(\mathbb{R}) \to L^2(\mathbb{R})$ is not Fredholm.

Bonus question 2. In Problem 3, show that the image of T is closed.