Lecture 3 The Seiberg-Witten equations.

\mathfrak(s)

From last time Spin^c structure $\mathfrak{s}$

on 4-mfld M is $S \to M$

$S = S^+ \oplus S^-$ $\quad \mathrm{rk}_{\mathbb{C}} S^{\pm} = 2$

Hermitian

$Cl(TM) \curvearrowright S$ Clifford mult

$TM \curvearrowright S$ via spinor representation.

$\exists$ ∇^S compatible connection

$$\nabla^S_X(Y \cdot s) = (\nabla_X Y) \cdot s + Y \cdot \nabla^S_X s.$$

$\forall s \in \mathcal{C}^\infty(S)$, X, Y vector fields.

we'll call this a spin^c connection.

A, ∇_A

Suppose ∇_A, $\nabla_{A'}$ are spin^c connections.

- $\nabla_A - \nabla_{A'} = \tilde{a} \in \Omega^1(\mathfrak{u}(S))$
  (skew hermitian connections)
  $\begin{pmatrix} ia & & \\ & \ddots & \\ & & ia \end{pmatrix}$

- $\forall X, Y, s$, compatibility $\Rightarrow$

$$\tilde{a}(X)\cdot(Y\cdot s) = Y\cdot(\tilde{a}(X)\cdot s).$$

$\Rightarrow \tilde{a}(X)$ commutes with Y.

Notice $Cl(\mathbb{R}^4) \curvearrowright \mathbb{C}^4$ is irreducible!

$\Rightarrow \tilde{a}(x)$ is a multiple of identity

by Schur's lemma.

$\Rightarrow \tilde{a} = a \otimes \mathbb{1}_S \quad a \in \Omega^1(i\mathbb{R})$.

<u>Lemma</u> {spin^c connections on S}

is an affine space over $\Omega^1(i\mathbb{R})$.

<u>**Def**</u> A spin^c connection on S ($\Rightarrow S^+, S^-$)

$A^t :=$ the connection induced on

$\det(S^+) = \Lambda^2 S^+$ $\rightarrow$ line bundle!

if $A - A' = a \otimes 1_S \qquad a \in \Omega^1(i\mathbb{R})$

$$\Rightarrow A^t - (A')^t = 2a \qquad \left(\det(\lambda \cdot A) = \lambda^2 \det A\right)$$

$F_{A^t} \in i\Omega^2$ imaginary valued 2-form.

<u>Now</u> $\overset{\dim = 6}{\Omega^2} = \overset{\dim 3}{\Omega^+} \oplus \overset{\dim 3}{\Omega^-}$

$\circlearrowleft$ $*$, $*^2 = 1$ $\quad \to e_1 \wedge e_2 + e_3 \wedge e_4$

Recall $\Omega^2 \subset \Lambda^* T^*M \otimes S$

via Clifford mult.

<u>Key computation</u> via this section,

$\Omega^{\pm}$ acts as elements in $su(S^{\pm})$

traceless skew hermitian

Self dual 2-form $\rightsquigarrow$ endomorphism of S^+.

$i\Omega^+$ acts as $i\,su(S^+)$

traceless hermitian.

SW equations (A, Φ)

spin^c connection

spinor $\in \mathcal{C}^\infty(S^+)$

Eq I A spin^c connection

$\rightsquigarrow D_A^+ : \mathcal{C}^\infty(S^+) \to \mathcal{C}^\infty(S^-)$.

$$\boxed{D_A^+ \Phi = 0} \quad \in \mathcal{C}^\infty(S^-).$$

Eq II $\quad \Phi \in \mathcal{C}^\infty(S^+)$

$\rightsquigarrow (\Phi\Phi^*)_0 \in \mathcal{C}^\infty(isu(S^+))$

$\Phi \rightsquigarrow \quad \Phi\Phi^* : S^+ \circlearrowleft$

$$v \longmapsto \langle v, \Phi \rangle \Phi .$$

$(\Phi\Phi^*)_0$ traceless part of this.

In coordinates, if $\Phi = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$

$$\Rightarrow (\Phi\Phi^*)_0 = \begin{pmatrix} \frac{1}{2}(|\alpha|^2 - |\beta|^2) & \alpha\bar{\beta} \\ \bar{\alpha}\beta & \frac{1}{2}(|\beta|^2 - |\alpha|^2) \end{pmatrix}.$$

$$\rho(F_A^{t^+}) \in \mathcal{C}^\infty(i\mathfrak{su}(S^+))$$

$$\boxed{\frac{1}{2}\rho(F_{A^t}^+) = (\Phi\Phi^*)_0} \in \mathcal{C}^\infty(i\mathfrak{su}(S^+))$$

(A, Φ)

$$\begin{cases} D_A^+ \Phi = 0 & \in \mathcal{C}^\infty(S^-) \\ \frac{1}{2}\rho(F_{A^t}^+) = (\Phi\Phi^*)_0 & \in \mathcal{C}^\infty(i\mathfrak{su}(S^+)) \end{cases}$$

<u>Seiberg - Witten eqns.</u>

<u>Gauge group</u> $\mathcal{G} = \mathrm{Maps}(M, \overset{U(1)}{\overset{\shortparallel}{S^1}})$

(automorphism of S as a <u>Clifford bundle</u>)

$$u\cdot(A,\Phi)=(A-u^{-1}du\,,\,u\cdot\Phi)$$

The SW are $\mathcal{G}$-invariant! (pset).

<u>Def</u> $\mathcal{C}(X,S)=\left\{(A,\Phi)\;\middle|\;\begin{array}{l}A \text{ spin}^c \text{ connection}\\ \Phi\in C^\infty(S^+)\end{array}\right\}$

configuration space.

$$\mathcal{B}(X,S)=\mathcal{C}(X,S)/\mathcal{G}$$

moduli space of configurations.

(will take L^2_k-completions).

<u>Stabilizers</u> for $\mathcal{G} \curvearrowright \mathcal{C}(X, \mathfrak{s})$.

$$u \cdot (A, \Phi) = (A - u^{-1}du, u \cdot \Phi)$$

$$\shortparallel$$

$$(A, \Phi) \qquad (u \in \mathrm{Stab}(A, \Phi))$$

$\Rightarrow A = A - u^{-1}du \Rightarrow du = 0 \Rightarrow u$ is constant.

$\Rightarrow u\Phi = \Phi$.

<u>Def</u> (A, Φ) is

- <u>irreducible</u> if $\Phi \not\equiv 0$ $(\mathrm{stab} = \{1\})$

- <u>reducible</u> $\Phi \equiv 0$ (stab $= S^1$)

constant transformation

$$\mathcal{B}(X,\mathfrak{s}) \supseteq \mathcal{B}^*(X,\mathfrak{s})$$ ← irreducible config.

$$\cup\!\!\mid$$

$\mathcal{M}(X,\mathfrak{s}) = \{\text{solutions to SW equns}\}/\mathcal{G}$.

moduli space of solutions.

<u>Thm</u> $\mathcal{M}(X,\mathfrak{s})$ is compact! ($\mathcal{C}^\infty$ top).

(we'll assume $b_1(X) = 0$).
for simplicity.

Gauge fixing A_0 base connection.

Def A is in Coulomb gauge (wrt A_0)

if $d^*\underbrace{(A^t - A_0^t)}_{\in \Omega^1(i\mathbb{R})} = 0$. $(= \mathrm{div} = 0)$.

Prop Every A is gauge equivalent to a unique connection in Coulomb gauge.

Proof $A - A_0 = a \otimes \mathbb{1}$

$$A^t - A_0^t = 2a \in \Omega^1(i\mathbb{R}).$$

$$(u \cdot A)^t - A_0^t = \underbrace{2a - 2u^{-1}du}_{d^* = 0}$$

$b_1 = 0$, $u : M \to S^1 \Rightarrow u = e^{\xi}$

for $\xi : M \to i\mathbb{R}$.

if $u = e^{\xi}$

$$u^{-1}du = e^{-\xi} \cdot (e^{\xi} \cdot d\xi) = d\xi$$

Want to solve

$$0 = d^*(a - u^{-1}du) = d^*(a - d\xi)$$

$$= d^*a - d^*d\xi \qquad (\xi \in \Omega^0 \Rightarrow d^*\xi = 0)$$

$$= d^*a - \Delta\xi$$

want

$$\Rightarrow \Delta\xi = d^*a$$

$\Rightarrow \xi$ exists by the Hodge theorem!

(unique up to constants).

$$\Big(\Omega^0 = \underbrace{\ker \Delta}_{=\,\mathbb{R}} \oplus \underbrace{d^* \Omega^1}_{=\,\mathrm{Im}\,\Delta} \Big).$$

<u>Fact</u> (Pset) $\quad d^* \oplus d^+$

$$\Omega^1 \longrightarrow \Omega^0 \oplus \Omega^+$$

is elliptic!

$$\begin{cases} D_A^+ \Phi = 0 \\[2em] \frac{1}{2} \rho(F_{A^t}^+) = (\Phi \Phi^*)_0 \end{cases}$$

SW + gauge fixing.

$$\left(d^*(A^t - A_0^t) = 0 \right.$$

is elliptic (non-linear!).

Key point A priori estimates for $|\Phi|$

Main input Weitzenböck formula.

$\nabla(A, \Phi)$

have same principal symbol.

$$\overbrace{D_A^- D_A^+ \Phi} = \overbrace{D_A^* D_A \Phi}$$

scalar curvature,

$$\underbrace{+ \frac{1}{2}\rho(F_{A^t}^+)\Phi + \frac{s}{4}\Phi}_{\text{0-th order}}$$

very geometric!

$\underline{\text{Now}}$ $\Delta|\Phi|^2 = 2\langle \Phi, \nabla_A^* \nabla_A \Phi \rangle - 2|\nabla_A \Phi|^2$

$$\left(-\frac{d^2}{dx^2} f^2 = 2 f \cdot \left(-\frac{d^2}{dx^2} f \right) - 2 \left(\frac{d}{dx} f \right)^2 \right).$$

$\leq 2\langle \Phi, \nabla_A^* \nabla_A \Phi \rangle =$ if (A, Φ) solves SW $D_A^+ \Phi = 0$

$$= -2\langle \Phi, \Big(\underbrace{\tfrac{1}{2} \rho(F_A^t)}_{(\Phi\Phi^*)_0} + \frac{s}{4} \Big) \Phi \rangle$$

$$= -|\Phi|^4 - \frac{1}{2} s |\Phi|^2.$$

If (A, Φ) is soln

$$\Rightarrow \Delta|\Phi|^2 \leq -|\Phi|^4 - \frac{1}{2} s |\Phi|^2.$$

If $|\Phi|^2$ is maximal at p

$\Rightarrow \Delta|\Phi|^2_{(p)} \geq 0$

$\Rightarrow |\Phi|^4(p) \leq -\frac{1}{2} s|\Phi|^2(p)$

CORRECTION:

$\Rightarrow$ at maximum, if $|\Phi|^2(p) > 0$.

$$|\Phi|^2_{(p)} \leq -\frac{1}{2} s(p) \leq -\frac{\inf s}{2}.$$

<u>Cor</u> If $\Phi \neq 0$. $|\Phi|^2 \leq -\frac{\inf(s)}{2}$ <u>everywhere</u>

a priori estimate!!

Proof of cptness of $\mathcal{M}(X, \mathfrak{s})$ (in L^2).

$$\begin{cases} D_A^+ \Phi = 0 \\ \frac{1}{2}\rho(F_{A^t}^+) = (\Phi\Phi^*)_0 \\ d^*(A^t - A_0^t) = 0. \end{cases}$$

we have $\|\Phi\|_{L^\infty} \leq C$

$\Rightarrow \|(\Phi\Phi^*)_0\|_{L^2} \leq C$ (diff constant.)

If $A^t - A_0^t = 2a$ $F_{A^t}^+ - F_{A_0^t}^+ = 2da^+$

II eq + gauge fixing

$$\|(d^+ \oplus d^*) a\|_{L^2} \leq C .$$

$\Rightarrow$ elliptic $d^+ \oplus d^*$ ($b_1 \Rightarrow$ no kernel!).

$\Rightarrow \|a\|_{L^2_1} \leq C$. (elliptic estimate).

$\Downarrow$

$$\|a\|_{L^4} \leq C$$

$\boxed{\text{I eq}}$ $D_A^+ \Phi = 0$

$$D_{A_0}^+ \Phi + \underset{L^4}{\underset{\cup}{\rho(a)}} \underset{L^4}{\underset{\cup}{\Phi}} = 0 .$$

$$\|D_{A_0}^+ \Phi\|_{L^2} \leq C$$

$\Rightarrow D^+_{A_0}$ elliptic

$\Rightarrow \|\Phi\|_{L^2_1} \le C$.

By Rellich, $L^2_1 \hookrightarrow L^2$ is cpt

$\Rightarrow \{\Phi_n\}$ ~~compact~~ precompact in L^2.

TO DO: show that (A_n, Φ_n) $\swarrow$

solution!