MSRI SUMMER SCHOOL 2021: AN INTRODUCTION TO SEIBERG-WITTEN INVARIANTS

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This minicourse will cover the foundational aspects of Seiberg-Witten theory on four-manifolds, with a specific emphasis on the analysis and differential geometry behind the constructions. Here is an outline of each lecture.

Lecture 1.

- Definition of elliptic operators. Principal symbols (both in coordinates and intrinsic). Basic examples.
- Sobolev spaces on manifolds and their basic properties (Sobolev embedding, Rellich)
- The elliptic estimate, the Fredholm property for elliptic problems, definition of the index.
- The Hodge star and Hodge operator. The Hodge Laplacian.
- Statement of the riemannian Hodge theorem, computation of the index of $d + d^*$.

Main references:

- Brezis - Functional analysis, Sobolev spaces and PDEs.
- Warner - Foundations of Differentiable Manifolds and Lie Groups, Ch.6.

Lecture 2.

- Brief historical introduction: Dirac's motivation for the square root of the Laplacian
- Definition of Clifford algebras and basic properties. Classification of complexified (even dimensional) Clifford algebras
- Clifford modules, the spinor representation, mostly in the explicit case of $n = 4$. Pauli matrices.
- Definition of Clifford bundle and the associated Dirac operators.
- Spin^c structures as a special case of Clifford bundles. Explicit example for $n = 2$. The spinor bundles splits in the chiral parts. The index of the chiral Dirac operators.
- Classification of spin^c structures on a 4-manifold (statement only).

Main references:

- Roe - Elliptic Operators, Topology, and Asymptotic Methods, Ch. 3, 4.

Lecture 3.

- The space of spin^c connections is an affine space over imaginary value one forms. Connections on the determinant line bundle.
- The unperturbed Seiberg-Witten equations.
- The action of the gauge group. Definition of the various configurations (uncompleted) spaces. Reducible and irreducibles.
- Statement of compactness theorem. The Coulomb gauge (case $b_1 = 0$).
- The Weitzeböck formula, a priori bounds. Proof of compactness (only in the L^2 norm).
- Brief discussion of the variational interpretation of the equations.

Main references:

- Morgan - The Seiberg-Witten Equations and Applications to the Topology of Smooth Four-Manifolds, Ch. 4, 5.

Lecture 4.

- The completed moduli space and its topology (case $b_1 = 0$).
- The trouble with reducibles. The perturbed equations. Existence of reducibles for the perturbed equations.
- Statement of the transversality theorem. Dimension of the moduli spaces. Sketch of proof. A couple of words on orientations.
- Definition of the invariants. Sketch of proof of invariance when $b^+ \geq 2$.
- Relation with positive scalar curvature. Statement of the simple type conjecture.

Main references:

- Morgan - The Seiberg-Witten Equations and Applications to the Topology of Smooth Four-Manifolds, Ch. 6.

Lecture 5.

- Statement and proof of the adjunction inequality in the case of self-intersection zero.
- The Thom conjecture. Basic strategy behind the proof via blow-up until get self-intersection zero, and discussion of the problems that arise ($b^+ = 1$, how to show non vanishing?)
- Statement of the wall crossing formula for $b_1 = 0$, and idea behind the proof.
- Computation of the invariants for blown-up projective plane using PSC and wall-crossing. Sketch of proof of Thom conjecture.
- Definition of positive and negative sides of the wall.

Main references:

- Kronheimer, Mrowka - Monopoles and three-manifolds, Ch. 40.
- Kronheimer, Mrowka - The genus of embedded surfaces in the projective plane.