ICTP SUMMER SCHOOL 2022: FAMILIES OF DIRAC OPERATORS AND APPLICATIONS

FRANCESCO LIN

Contact. You can reach me at flin@math.columbia.edu.

Materials. I will post notes for the lectures on my website; they will contain some exercises for each lecture.

Teaching assistant. Juan Muñoz-Echániz.

Contents

The goal of this minicourse is to introduce the general ideas of index theory, and then discuss the very concrete case of families of Dirac operators and applications to geometry and topology. Here is a rough description of the content of each lecture; for the prerequisites, you should take them as a list of things that you have heard about before at least once - feel free to reach out in case you want some reference.

Lecture 1. An informal introduction to spin Dirac operators. Fredholm operators and their index. The Lichnerowicz formula. The Atiyah-Singer index theorem and applications to metrics of positive scalar curvature. Families of twisted Dirac operators.

References: Roe, Elliptic operators, asymptotic methods and topology.

Prerequisites: Pontryagin classes; connections on vector bundles and their curvature; definition of scalar curvature of a Riemannian manifold, basic definitions in functional analysis such as separable Hilbert space, bounded operator.

Lecture 2. Introduction to K-theory, and how it relates to Fredholm operators (the Atiyah-Jänich theorem); the Chern character and the index formula. Metrics of positive scalar curvature on the torus.

References: Atiyah, K-theory; Gromov-Lawson, Spin and Scalar Curvature in the Presence of a Fundamental Group. I.

Prerequisites: complex vector bundles and their classifying spaces; Chern classes.

Lecture 3. The space of self-adjoint Fredholm operators. Coupled Morse theory and applications to monopole Floer homology and the 3D Weinstein conjecture.

References: Kronheimer-Mrowka, Monopoles and three-manifolds, Ch. 33, 35. Hutchings, Taubes' proof of the Weinstein conjecture in dimension three.

Prerequisites: Morse homology; having a vague idea of either HF^∞ or $\overline{HM}$.