Calculus I
Section 005
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**Practice Midterm I Answers**

The real midterm will be of about this difficulty and length. While I have not given point values here, on the actual exam, all of these would be given roughly the same values, except for question 2, which would be worth twice as much.

1. Let $f(x)$ be a function such that $\lim_{x\to 2} f(x) = 3$.

a. Can it happen that f(1.999999) = 1700? Explain.

b. Can it happen that f(2) = -23? Explain.

c. If we are also given that $f(x)$ is continuous at $x = 2$, how will your answers to parts a and b change?

**Answers: a.** Yes, this can happen, as long as between 1.999999 and 2 the function dips really fast.

**b.** Yes; for example, consider $f(x) = \begin{cases} 3 & \text{if} \quad x \neq 2 \\ -23 & \text{if} \quad x = 2. \end{cases}$

**c.** $f(1.999999)$ can still be 1700, but continuity means that $\lim_{x\to 2} f(x) = f(2) = 3$, so $f(2)$ cannot be -23.

2. Evaluate the following limits, or state that they do not exist. If there is an infinite limit ($\infty$ or $-\infty$), indicate that as well. Show work and/or give explanations for your answers.

a. $\lim_{x \to 3} \sqrt{x+6}$

b. $\lim_{x \to \frac{\pi}{3}} \cos x$

c. $\lim_{t \to 3} \frac{t^2 - 9}{t^2 - 4t + 3}$

d. $\lim_{y \to -5} \frac{|y+5|}{y+5}$ (Hint: $\frac{|y+5|}{y+5}$ is easily expressed as a piecewise-defined function; if you are unsure how to do that, try plugging in a few values for $y$.)

e. $\lim_{t \to 1^-} \frac{t-3}{t^2 - 4t + 3}$

f. $\lim_{t \to 0} \frac{1}{3t} - \frac{1}{t\sqrt{9+t}}$

**Answers:** **a.** 3 (direct substitution)

**b.** $\frac{1}{2}$ (direct substitution)

**c.** 3 (factor top and bottom, cancel, substitute)

**d.** Limit does not exist, no infinite limit (function approaches -1 from the left, +1 from the right)

**e.** Limit does not exist, as $\lim_{t \to 1^-} \frac{t-3}{t^2 - 4t + 3} = -\infty$ (factor bottom and cancel; as $t$ goes to 1 from the left, $t - 1$ goes to 0 from the **left**=negative side)

**f.** $\frac{1}{54}$ (get a common denominator, apply the "conjugate trick", factor out some $t$'s)

3. At time $x$ hours after noon, the number of Sharks on a street is given by $2^x + 3$, while the number of Jets on the same street is given by $x^3 + 2$. If there are ever the same number of Jets and Sharks on the street, there will be a rumble. Show that there will be a rumble sometime between 1PM and 2PM. (Assume that fractions of gang members are possible.)

Hint: we need to find when $2^x + 3 = x^3 + 2$; rephrase this equation in terms of a function $f(x)$. This function should be continuous, and you will need that fact.

**Solution:** Set $f(x) = 2^x + 3 - (x^3 + 2)$; we need to show that $f(x) = 0$ has a solution for some value $x$ between 1 and 2. To show this we use the Intermediate Value Theorem. We can use the Intermediate Value Theorem for the function $f(x)$ because it is continuous (since it is a sum of functions that we know are continuous). Check $f(1) = 2$ and $f(2) = -3$; since 0 is between 2 and -3, IVT says that there is some $x$ between 1 and 2 with $f(x) = 0$. So for that $x$, $2^x + 3 = x^3 + 2$, and there will be a rumble.

4. Let $f(x) = \begin{cases} 2x^2 - 5x + 16 & \text{if} \quad x < 0; \\ \frac{16}{1-x} & \text{if} \quad 0 \leq x < 5; \\ 8 & \text{if} \quad x \geq 5. \end{cases}$

a. At what values of $x$ is $f(x)$ discontinuous?

b. Compute $f'(-1)$ (Hint: the derivative at $x = -1$ only depends on how $f(x)$ behaves near $x = -1$; near $x = -1$, what (simpler) function does $f(x)$ behave like?)

**Answers: a.** $x = 1$ and $x = 5$: among the components of the piecewise-defined functions, the only place where any of them will be discontinuous will be the middle one, at $x = 1$. Aside from that, we only have to check to see if our function is continuous at $x = 0$ and $x = 5$. At $x = 0$ $\lim_{x\to 0} f(x) = f(0)$ (the limit exists because the righthand and lefthand limits are equal), so $f(x)$ is continuous there. At $x = 5$, however, $\lim_{x\to 5^-} f(x) = -4$, while $\lim_{x\to 5^+} f(x) = 8$, so $\lim_{x\to 5} f(x)$ does not exist, and thus $f(x)$ is not continuous there.

**b.** -9: $f(x) = 2x^2 - 5x + 16$ for values of $x$ near -1; so $f'(x) = 4x - 5$ for these values (using the derivative rules), and so $f'(-1) = -9$.

5. a. Let $f(x) = \dfrac{1}{\sqrt{x+3}}$. Using the definition of the derivative, find $f'(x)$.

b. Find the equation of the tangent to the graph of $y = f(x)$ at the point $(1, \frac{1}{2})$.

**Answers: a.** $f'(x) = \frac{-1}{2(x+3)\sqrt{x+3}}$: use the second definition of the derivative – the one where $h \to 0$ – and evaluate:

$$\lim_{h\to 0} \frac{\frac{1}{\sqrt{(x+h)+3}} - \frac{1}{\sqrt{x+3}}}{h}$$

$$= \lim_{h\to 0} \frac{\sqrt{x+3} - \sqrt{(x+h)+3}}{h\sqrt{(x+h)+3}\sqrt{x+3}}$$

$$= \lim_{h\to 0} \frac{\sqrt{x+3} - \sqrt{(x+h)+3}}{h\sqrt{(x+h)+3}\sqrt{x+3}} \left(\frac{\sqrt{x+3} + \sqrt{(x+h)+3}}{\sqrt{x+3} + \sqrt{(x+h)+3}}\right)$$

$$= \lim_{h\to 0} \frac{x+3-((x+h)+3)}{h\sqrt{(x+h)+3}\sqrt{x+3}(\sqrt{x+3} + \sqrt{(x+h)+3})}$$

$$= \lim_{h\to 0} \frac{-h}{h\sqrt{(x+h)+3}\sqrt{x+3}(\sqrt{x+3} + \sqrt{(x+h)+3})}$$

$$= \lim_{h\to 0} \frac{-1}{\sqrt{(x+h)+3}\sqrt{x+3}(\sqrt{x+3} + \sqrt{(x+h)+3})}$$

$$= \frac{-1}{\sqrt{x+3}\sqrt{x+3}(\sqrt{x+3}+\sqrt{x+3})}$$

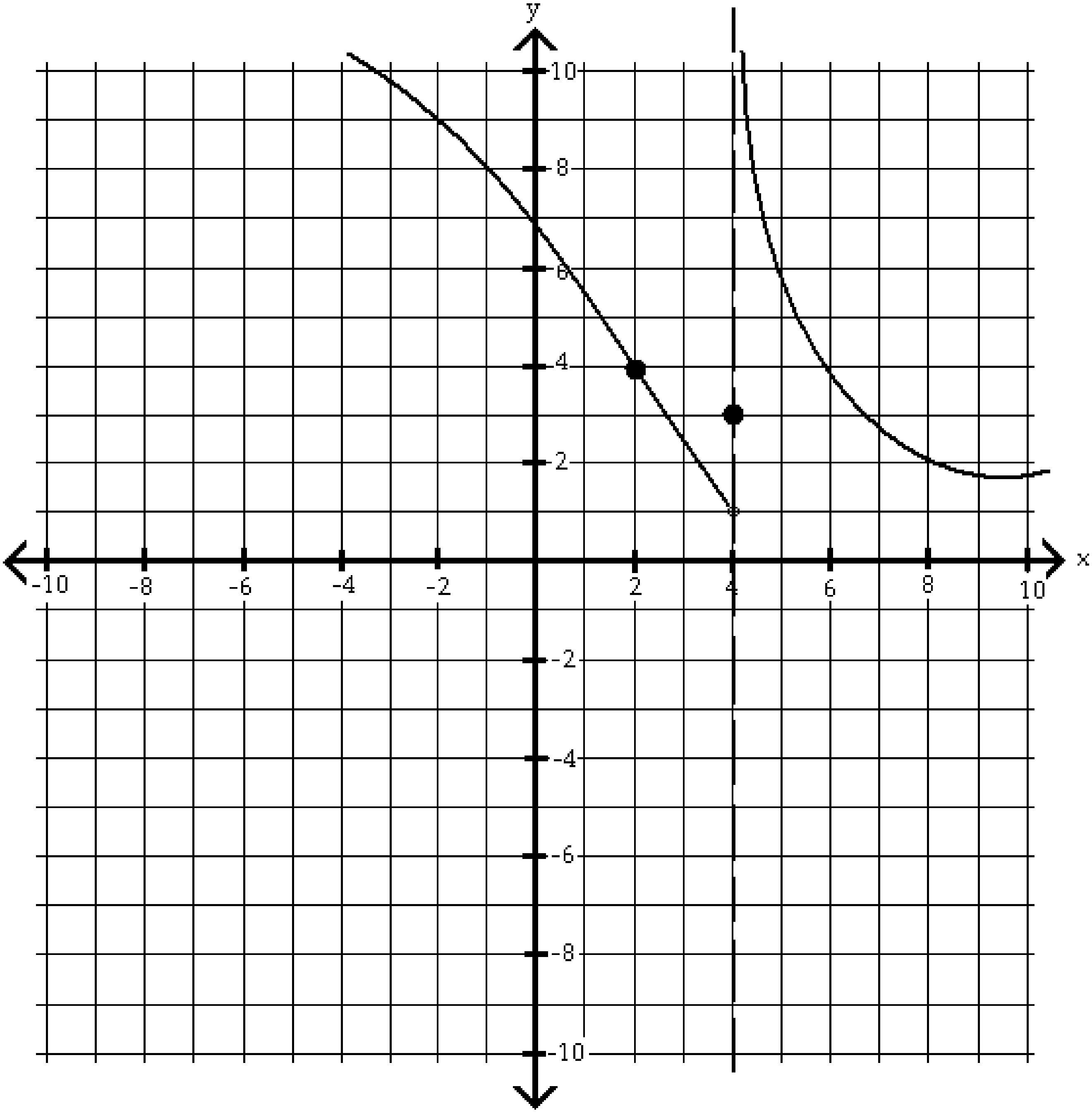
$$= \frac{-1}{2(x+3)\sqrt{x+3}}$$

**b.** $y - \frac{1}{2} = \frac{-1}{16}(x-1)$

6. Sketch the graph of a function that satisfies **all** of the following:

- $\lim_{x \to 2^-} f(x) = 4$
- $f(x)$ is continuous at $x = 2$
- $\lim_{x \to 4^-} f(x) = 1$
- $\lim_{x \to 4^+} f(x) = -\infty$
- $f(4) = 3$

**Answer:** An example is given by the graph on the next page.



y
x
10
8
6
4
2
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-2
2
4
6
8
10