1.4 Fans and toric varieties

First, some conventions. I will be using the same setup as in my last talk (i.e. the setup that Fulton uses throughout): this means, for instance, that things like $N$, $N_R$, $n$, and $\langle , \rangle$ are throughout assumed to be given. (For more information on the precise setup and definitions here, see the my notes from my last talk.) From now on, any cone that we consider is assumed to be a strongly convex rational polyhedral cone unless otherwise stated. Also, Fulton occasionally makes comments about concepts from algebraic geometry that we didn’t really discuss. I will express such comments in brackets in my notes. If you’re someone who knows what the ideas in brackets mean, then hopefully they’ll help illuminate some of the underlying machinery; if you aren’t, however, don’t worry — we won’t actually be working with these concepts.

We begin by defining fans, another key object in our discussion of cones and toric varieties.

Definition 1. A fan in $N$ is a set $\Delta$ of strongly convex rational polyhedral cones in $N_R$ such that

1. Every face of a cone in $\Delta$ is also in $\Delta$; and

2. The intersection of two cones in $\Delta$ is a face of each cone (and hence must lie in $\Delta$).

We will assume unless otherwise stated that all fans are finite, i.e. that they contain a finite number of cones. [This assumption means that the toric varieties we construct from fans below will be of finite type, not just locally of finite type].

Now, given a fan $\Delta$, we know that given any two cones $\sigma$ and $\tau$ in $\Delta$, their intersection $\sigma \cap \tau$ is a face of each. As Greyson discussed last time, this means that $U_{\sigma \cap \tau}$ is isomorphic (as a variety) to a principal open
subvariety of both $U_\sigma$ and $U_\tau$. As Mitchell discussed, given an isomorphism of subvarieties of two varieties, we can glue the original varieties into a new variety by identifying them along the isomorphic subvarieties. In this case, we can glue $U_\sigma$ and $U_\tau$ into one variety. In fact, we can do this repeatedly, gluing all of the cones in $\Delta$. We call the resulting variety a toric variety, and we denote it $X(\Delta)$.

There is one technical detail I’ve glossed over in the above description. When gluing more than 2 varieties together, one has to check that the different gluings are compatible. If $\varphi_{\sigma\tau}$ is the isomorphism on $U_{\sigma\cap\tau}$, then the compatibility condition is that, for any $\sigma$, $\sigma'$, and $\tau$ in $\Delta$, we have

$$\varphi_{\sigma\tau} = \varphi_{\sigma\sigma'} \circ \varphi_{\sigma'\tau}$$

on the open where this statement makes sense (i.e. on $U_{\sigma\cap\tau} \cap U_{\sigma'\cap\tau} = U_{\sigma\cap\sigma'\cap\tau}$). Conceptually, this condition ensures that, if we use $\varphi_{\sigma'\tau}$ to glue $U_{\sigma'}$ to $U_\tau$ and then use $\varphi_{\sigma\sigma'}$ to glue this to $U_\sigma$, then we get the same thing as if we first use $\varphi_{\sigma\tau}$ to glue $U_\sigma$ to $U_\tau$ and then glue $U_{\sigma'}$ to this. Without such a condition, there is really no canonical structure to put on the gluing of all these varieties, since the structure would depend on the order of the gluing. In our case, however, we know that the coordinate ring of $U_{\sigma\cap\tau} \subseteq U_\sigma$ is exactly the same as the coordinate ring of $U_{\sigma\cap\tau} \subseteq U_\tau$, so $\varphi_{\sigma\tau}$ is the isomorphism coming from the identity map on this coordinate ring. From this, one can check that the compatibility condition is satisfied, essentially because the composition of identity maps is again the identity map. [To be rigorous, we need to use the fact that for any ring $R$ and any multiplicative subset $S$, the map $S^{-1}R \to S^{-1}R$ induced by the identity map $R \to R$ is the identity map.]

Mitchell defined in his notes (but did not get to in class) the notion of a separated variety, which is essentially an analog of Hausdorff for topological spaces. One can check that for any fan $\Delta$, $X(\Delta)$ is separated. [The proof is as follows. Note that $\{U_\sigma : \sigma \in \Delta\}$ is an affine cover of $X(\Delta)$. We know that every intersection $U_\sigma \cap U_\tau$ is equal to $U_{\sigma\cap\tau}$ and hence is affine. Moreover, for any $\sigma$ and $\tau$ in $\Delta$, we know that $S_{\sigma\cap\tau} = S_\sigma + S_\tau$, which implies that $A_\sigma \otimes A_\tau \to A_{\sigma\cap\tau}$ is surjective. By a standard equivalent condition to separatedness, this proves that $X(\Delta)$ is separated.]

We now give a few examples to help make concrete the connection between fans and toric varieties.

**Example.** Let $\sigma$ be any cone, and define $\Delta$ to be the fan consisting of $\sigma$ along with all of its faces. Then, all of our gluings will just be embeddings of $U_\tau$ into $U_\sigma$ for all the faces $\tau$ of $\sigma$. Thus, $X(\Delta) \cong U_\sigma$ is an affine toric variety. (In fact, it turns out that this is the only way to get a toric variety which is affine.)

**Remark.** The reasoning in the above example can be generalized slightly. Given any cone $\tau$ in a fan $\Delta$, if $\tau$ is the proper face of some other cone $\sigma$
This fact is summarized by the following proposition.

**Remark.** Notice that whenever we have a fan $\Delta$ containing some maximal cone $\sigma$ which is disjoint from every other maximal cone of $\Delta$, the gluing of $U_\sigma$ to construct $X(\Delta)$ will all be trivial, and we will just get

$$X(\Delta) = U_\sigma \sqcup X(\Delta - \{\sigma\}).$$

For this reason, we usually consider fans in which no such $\sigma$ exists.

**Example.** (Classifying toric varieties when $\dim N = 1$). Let $N \cong \mathbb{Z}$, and let $\sigma_+ = \mathbb{R}_{\geq 0}$, $\sigma_- = \mathbb{R}_{\leq 0}$, and $\sigma_0 = \{0\}$. There are four interesting fans to consider over $N$: $\Delta_0 = \{\sigma_0\}$, $\Delta_+ = \{\sigma_+, \sigma_0\}$, $\Delta_- = \{\sigma_-, \sigma_0\}$, and $\Delta_\pm = \{\sigma_+, \sigma_-, \sigma_0\}$. (Any other fan either gives rise to a variety isomorphic to the one coming from these or else gives the disjoint union of two varieties, as discussed in the above remark.) Notice that $\sigma_+$ and $\sigma_-$ are self-dual, while $\sigma_0' = \mathbb{R}$ is generated by $\pm 1$. So, $A_{\sigma_+} = \mathbb{C}[x]$, $A_{\sigma_-} = \mathbb{C}[x^{-1}]$, and $A_{\sigma_0} = \mathbb{C}[x, x^{-1}]$. By the above example, we immediately get:

$$X(\Delta_0) = U_{\sigma_0} \cong \mathbb{C}^\times;$$

$$X(\Delta_+) = U_{\sigma_+} \cong \mathbb{C};$$

and

$$X(\Delta_-) = U_{\sigma_-} \cong \mathbb{C}.$$

As for $X(\Delta_\pm)$, we need to glue $U_{\sigma_+}$ and $U_{\sigma_-}$ along $U_{\sigma_0}$. On the level of ring homomorphisms, the embedding $U_{\sigma_0} \hookrightarrow U_{\sigma_+}$ corresponds to $\mathbb{C}[x] \hookrightarrow \mathbb{C}[x]_x = \mathbb{C}[x, x^{-1}]$, while $U_{\sigma_0} \hookrightarrow U_{\sigma_-}$ corresponds to $\mathbb{C}[x^{-1}] \hookrightarrow \mathbb{C}[x^{-1}]_{x^{-1}} = \mathbb{C}[x, x^{-1}]$. Thus, our isomorphism on principal open sets corresponds to the ring homomorphism $\mathbb{C}[x, x^{-1}]$ sending $x \mapsto x = (x^{-1})^{-1}$. As we’ve discussed before, this variety is 1-dimensional projective space $\mathbb{P}^1$.

**Example.** With $N \cong \mathbb{Z}^2$, define $\sigma_+$ to be the cone generated by $(1, 0)$ and $(0, 1)$ and $\sigma_-$ to be the cone generated by $(-1, 0)$ and $(0, 1)$. Let $\Delta$ be the fan consisting of $\sigma_+$ and $\sigma_-$, along with all the faces of these cones. Then, to get $X(\Delta)$, we glue $U_{\sigma_+} = \text{Spec} \, \mathbb{C}[x, y]$ and $U_{\sigma_-} = \text{Spec} \, \mathbb{C}[x^{-1}, y]$ along the intersection $U_{\sigma_+ \cap \sigma_-} = \text{Spec} \, \mathbb{C}[x, x^{-1}, y]$. Notice that the gluing here corresponds to the identity homomorphism on $\mathbb{C}[x, x^{-1}, y]$ which gives $\mathbb{P}^1 \times \mathbb{C}$.

This last example is actually an instance of a more general fact about toric varieties. This fact is summarized by the following proposition.
Proposition 2. Let $\Delta$ be a fan in a lattice $N$, and let $\Delta'$ be a fan in a lattice $N'$. Then, we define

$$\Delta \times \Delta' = \{ \sigma \times \sigma' : \sigma \in \Delta, \sigma' \in \Delta' \}.$$ 

$\Delta \times \Delta'$ is a fan in $N \oplus N'$. Moreover,

$$X(\Delta \times \Delta') \cong X(\Delta) \times X(\Delta').$$

Proof. Let $\sigma \in \Delta$ and $\sigma' \in \Delta'$ be cones with generating sets $\{v_i\}$ and $\{v'_i\}$, respectively. Then, the cone $\sigma \times \sigma'$ is generated by the images of all the $v_i$ and all the $v'_i$ in $(N \oplus N')_R \cong N_R \oplus N'_R$. From this, one sees that intersections of $\sigma \times \sigma'$ with other elements of $\Delta$ will be products of faces of $\sigma$ with those of $\sigma'$, which implies that $\Delta \times \Delta'$ is a fan. Moreover, since $\langle v_i, v'_j \rangle = 0$ for all $i$ and $j$ (in words: the $v_i$ live in the subspace $N_R$, which is orthogonal to the subspace $N'_R$ containing the $v'_j$), one sees that $(\sigma \times \sigma')^\vee = \sigma^\vee \times (\sigma')^\vee$, whence $S_{\sigma \times \sigma'} = S_\sigma \oplus S_{\sigma'}$ and $A_{\sigma \times \sigma'} \cong A_\sigma \otimes \mathbb{C} A_{\sigma'}$.

Notice that, by our above remarks and by definition of $X(\Delta \times \Delta')$, an open cover for this variety is given by $\{U_{\sigma \times \sigma'} \cong \text{Spec}(A_\sigma \otimes A_{\sigma'})\}$. On the other hand, the set $\{U_\sigma \times U_{\sigma'}\}$ is an open cover of $X(\Delta) \times X(\Delta')$. To give an isomorphism between $X(\Delta \times \Delta')$ and $X(\Delta) \times X(\Delta')$, it suffices to give isomorphisms between elements of these open covers. [Technically, we also need to worry about whether our local isomorphisms are compatible on intersections, but as usual, this is basically just a formal check.]. A fact (which is in Mitchell’s notes but which he did not get to in his lecture) is that, for any two varieties $X$ and $Y$, we have

$$R(X \times Y) \cong R(X) \otimes \mathbb{C} R(Y).$$

This implies that $U_\sigma \times U_{\sigma'} \cong \text{Spec}(A_\sigma \otimes A_{\sigma'}) \cong U_{\sigma \times \sigma'}$, which gives us isomorphisms between elements of our open covers, as desired. $\square$

Greyson showed us last week that the torus $T_N$ acts on any affine toric variety $U_\sigma$, i.e. we have a map $T_n \times U_\sigma \to U_\sigma$. It turns out that, when gluing varieties $U_\sigma$ to get $X(\Delta)$ for some fan $\Delta$, the isomorphisms along which we glue respect the toric action, so that it glues to give us a global action

$$T_N \times X(\Delta) \to X(\Delta)$$

for any $\Delta$. Moreover, this global action still extends the action of the torus on itself (here we are viewing $T_N$ as sitting inside of $X(\Delta)$ via its inclusion into any of the affine toric varieties we glued to define $X(\Delta)$, i.e. via $T_N \hookrightarrow U_\sigma$ for any $\sigma \in X(\Delta)$).

It turns out that this property uniquely characterizes toric varieties: that is, given any variety $X$ which has some dense open subvariety isomorphic to $T_N$, along with an action $T_n \times X \to X$ of the torus which extends
the action of the torus on itself, we have

\[ X \cong X(\Delta) \]

for a unique fan \( \Delta \) in \( N \). [Here we technically need \( X \) to be separated and normal.]

### 1.5 Toric varieties from polytopes

We begin with a little background from the theory of convex sets.

**Definition 3.** Given a subset \( I \) of \( \mathbb{R}^n \), we define the **convex hull** of \( I \), denoted \( \text{Conv} I \), to be the minimal convex set containing \( I \), or equivalently, the intersection of all convex sets containing \( I \).

The following fundamental result establishes an explicit expression for the convex hull of a set.

**Theorem 4.** Let \( I \) be a subset of \( \mathbb{R}^n \). Then,

\[ \text{Conv} I = \{ \sum_{i=1}^{r} a_i x_i : a_i \geq 0, x_i \in I, \sum_{i=1}^{r} a_i = 1 \} \]

**Definition 5.** We call a sum \( \sum_{i=1}^{r} a_i x_i \) such that \( \sum_{i=1}^{r} a_i = 1 \) a **convex combination** of the \( x_i \). Thus, an equivalent formulation of the above theorem is that \( \text{Conv} I \) is the set all convex combinations of elements of \( I \).

There isn’t so much to be said about convex sets in general, so we generally talk about convex objects with a little more structure. We now define such an object.

**Definition 6.** A **convex polytope** \( K \) in a finite-dimensional vector space \( V \) is the convex hull of a finite set of points in \( V \). A **face** \( F \) of \( K \) is the intersection of \( K \) with a supporting hyperplane of \( K \), or equivalently, a set of the form

\[ F = \{ v \in K : \langle u, v \rangle = r \} \]

where \( u \in V^* \) is a linear functional such that \( \langle u, v \rangle \geq r \) for all \( v \in K \) and \( r \) is any scalar. A **proper face** of \( K \) is a face not equal to \( K \). A **facet** of \( K \) is a face of codimension 1 (in the linear subspace spanned by elements of \( K \)).

**Remark.** Notice that, in the above definition, if we take \( u = 0 \), we get that \( F = K \). Thus, a convex polytope is always a face of itself.
Remark. Throughout, we will assume that convex polytopes are \(n\)-dimensional (i.e. that their elements span the vector space they lie in) and that they contain the origin in their interiors. In particular, this implies that facets are faces of dimension \(n - 1\) in the ambient vector space.

The following lemma provides a useful characterization of faces of convex polytopes.

\textbf{Lemma 7.} Let \(K = \text{Conv } X\) be a convex polytope (here \(X\) is a finite subset of a vector space \(V\)), and let \(F\) be a face of \(K\). Then, \(F\) is the set of all convex combinations of elements of \(X \setminus F\).

\textit{Proof.} Let \(F = \{v \in K : \langle u, v \rangle = r\}\), where \(u \in V^*\) and \(\langle u, v \rangle \geq 0\) for all \(v\) in \(K\). Then, any element \(v\) of \(K\) can be written as a convex combination \(\sum_{i=1}^m a_i x_i\), where \(x_i \in X\). So,

\[\langle u, v \rangle = \sum_{i=1}^m a_i \langle u, x_i \rangle.\]

By definition, \(\langle u, x_i \rangle \geq r\) for all \(i\). So, using the fact that \(\sum_{i=1}^m a_i = 1\), one sees that the above sum is equal to \(r\) precisely when \(\langle u, x_i \rangle = r\) for all \(i\), which implies the desired result. \(\square\)

\textbf{Remark.} In particular, the above lemma implies that every face of a convex polytope is itself a convex polytope, and moreover, that every convex polytope has finitely many faces (since there are finitely many subsets of the finite set \(X\) in the statement of the lemma).

It turns out that convex polytopes are closely related to cones, which is what motivates our discussion of them.

\textbf{Definition 8.} Let \(C\) be a convex subset of a vector space \(V\). Then, we define the \textit{cone over} \(C\) by

\[\sigma_C = \{\alpha \cdot c : \alpha \in \mathbb{R}_{\geq 0}, c \in C\} \subseteq V.\]

Now, suppose \(K = \text{Conv } X\) is a convex polytope. Notice that \(K \times \{1\}\) is the convex hull of the set \(X \times \{1\}\). Then, the cone over \(K \times \{1\}, \sigma_{K \times \{1\}}\), is a strongly convex polyhedral cone generated by \(X \times \{1\}\) in \(V \times \mathbb{R}\). To get a better geometric picture of this cone, we note that the cross-section of \(\sigma_{K \times \{1\}}\) with final coordinate \(r\) is a scaling of \(K\) by a factor of \(r\) sitting inside of \(V \times \{r\}\).

We would like to understand the dual cone \(\sigma_{K \times \{1\}}^\vee\). To this end, notice that \(u \times r \in V^* \times \mathbb{R}_{\geq 0}\) is in \(\sigma_{K \times \{1\}}^\vee\) if and only if \(\langle u \times r, v \rangle \geq 0\) for all \(v \in \sigma_{K \times \{1\}}\). We can then write \(v\) uniquely as \(\alpha(v' \times 1)\), where \(\alpha\) is some nonnegative real number and \(v' \in K\). So, we have

\[\langle u \times r, v \rangle = \alpha(\langle u, v' \rangle + r \cdot 1) \geq 0,\]
i.e. \( \langle u, v \rangle \geq -r \). Thus, \( u \times r \in \sigma_{K \times \{1\}}^V \) if and only if \( F = \{ v \in K : \langle u, v \rangle = -r \} \) is a face of \( K \). In this case, \( F \) consists of convex combinations of \( X \cap F \) by the above lemma, while the face \( \sigma_{K \times \{1\}} \cap u^+ \) is the cone generated by \( X \times \{1\} \cap u^+ \). But by the definition of \( F \), \( x \in X \cap F \) if and only if \( x \times 1 \in X \times \{1\} \cap u^+ \). So, \( \sigma_{K \times \{1\}} \cap u^+ \) is the cone over the convex polytope \( F \times 1 \). This proves that the faces of \( \sigma_{K \times \{1\}} \) are precisely the cones over the faces of \( K \). Moreover, this bijection between these faces preserves their codimension, so that the bijection restricts to a bijection between facets of \( \sigma_{K \times \{1\}} \) and facets of \( K \).

All of this nice structure allows us to translate several basic properties of the faces of cones into the world of convex polytopes, as summarized by the following proposition.

**Proposition 9.** Let \( K \) be a convex polytope in a vector space \( V \). Then,

1. Any face of a face of \( K \) is a face of \( K \);
2. The intersection of any two faces of \( K \) is a face of \( K \);
3. Any proper face of \( K \) is contained in a facet of \( K \);
4. Any proper face of \( K \) is the intersection of the facets of \( K \) which contain it; and
5. The topological boundary of \( K \) is the union of its proper faces (or equivalently, of its facets).

**Proof.** Points 1-4 follow immediately from taking the faces of \( K \) to the corresponding faces of \( \sigma_{K \times \{1\}} \) and applying the analogous properties about cones (see Proposition 8 in my previous notes, or points 3-6 in section 1.2 of Fulton). For the last point, notice that since \( K \times \{1\} \) is a cross-section of \( \sigma_{K \times \{1\}} \), it follows that the boundary of \( \sigma_{K \times \{1\}} \) intersected with \( V \times \{1\} \) is precisely the boundary of \( K \times \{1\} \). Likewise, the intersection of the proper faces of \( \sigma_{K \times \{1\}} \) with \( V \times \{1\} \) yields precisely the proper faces of \( K \times \{1\} \). Now, we know that the boundary of \( \sigma_{K \times \{1\}} \) is the union of the proper faces of \( \sigma_{K \times \{1\}} \), so intersecting these objects with \( V \times \{1\} \) gives us the desired result. \( \Box \)

Given our connection between cones and convex polytopes, we might expect to have a theory of duality of convex polytopes corresponding to the one in the case of cones. This is, in fact, the case.

**Definition 10.** Let \( K \) be a convex polytope in a vector space \( V \). Then, we define the polar set (or polar) of \( K \) by

\[
K^0 = \{ u \in V^* : \langle u, v \rangle \geq -1 \text{ for all } v \in K \}.
\]
Given any \( u \times r \in V^* \times \mathbb{R} \), we can write \( u \times r \) uniquely as \( \alpha(u' \times 1) \), where \( \alpha \) is some nonnegative real number, and for all \( v \in \sigma_{K \times \{1\}} \), we can likewise write \( v = \beta(v' \times 1) \), where \( v' \in K \) and \( \beta \in \mathbb{R}_{\geq 0} \). Then, the statement that \( u \times r \in \sigma_{K \times \{1\}}^\vee \) says that
\[
\langle u \times r, v \rangle = \alpha \beta(\langle u', v' \rangle + 1) \geq 0,
\]
or equivalently, \( \langle u', v' \rangle \geq -1 \) for all \( v' \in K \). This implies that \( u' \in K^0 \). So, \( \sigma_{K \times \{1\}}^\vee \) is the cone over \( K^0 \times \{1\} \) in \( V^* \times \mathbb{R} \). This allows us to translate the duality theory of cones into that of convex polytopes, which results in the following theorem.

**Theorem 11.** Let \( K \) be a convex polytope.

1. \( K^0 \) is a convex polytope, and \( (K^0)^0 = K \).

2. For any face \( F \) of \( K \), define
   \[ F^* = \{ u \in K^0 : \langle u, v \rangle = -1 \text{ for all } v \in F \}. \]
   \( F^* \) is a face of \( K^0 \), and the map \( F \mapsto F^* \) is a one-to-one and order-reversing correspondence between the faces of \( K \) and the faces of \( K^0 \).

3. For any face \( F \) of \( K \), \( \dim F + \dim F^* = \dim V - 1 \).

4. If \( K \) is rational (i.e. if its vertices lie in a lattice \( N \) of \( V \)), then \( K^0 \) is also rational, with vertices lying in \( N^\vee \).

**Sketch of proof.**

1. As noted about, \( \sigma_{K \times \{1\}}^\vee \) is the cone over \( K^0 \times \{1\} \). Given any generating set of \( \sigma_{K \times \{1\}}^\vee \), we can always scale each of the vectors in it by some nonnegative numbers and so take our generating set to be of the form \( X \times \{1\} \), where \( X \subset V \) is a finite subset. Then, one can check using definitions that \( K^0 = \text{Conv } X \), so that \( K^0 \) is a convex polytope. Moreover, \( (\sigma_{K \times \{1\}}^\vee)^\vee = \sigma_{K \times \{1\}} \) is the cone over \( (K^0)^0 \times \{1\} \); but \( \sigma_{K \times \{1\}} \) is also the cone over \( K \times \{1\} \), so we must have \( K = (K^0)^0 \).

2. If \( F \) is a face of \( K \), suppose that \( \tau \) is the cone over \( F \times \{1\} \). Then, essentially the same argument to show that \( \sigma_{K \times \{1\}}^\vee \) is the cone over \( K^0 \times \{1\} \) shows that the dual face \( \tau^* = \sigma_{K \times \{1\}} \cap \tau^\perp \) is the cone over \( F^* \times \{1\} \). Since \( \tau^* \) is a face of \( \sigma_{K \times \{1\}}^\vee \), this implies that \( F^* \) is a face of \( K^0 \). That the map is one-to-one and order-reversing comes from the same properties on the map \( \tau \mapsto \tau^* \) from faces of \( \sigma_{K \times \{1\}} \) to faces of \( \sigma_{K \times \{1\}}^\vee \).
3. Any face of $\sigma_{K \times \{1\}}$ satisfies the same property, and as noted above, the correspondence between faces of $K$ and faces of $\sigma_{K \times \{1\}}$ preserves dimension, so we immediately get the desired result.

4. Notice that $K = \text{Conv } X$, where $X$ is the set of vertices of $K$. So, if $X$ lies in a lattice $N$, then $X \times \{1\}$, which is a generating set of $\sigma_{K \times \{1\}}$, lies in the lattice $N \times \{1\}$, so $\sigma_{K \times \{1\}}$ is rational. This implies that $\sigma_{K \times \{1\}}^*$ is rational, and going from generators $\sigma_{K \times \{1\}}^*$ to expressing $K^0$ as a convex hull as in the proof of 1 above shows us that $K^0$ is rational.

Let $K$ be a rational convex polytope, and let $K'$ be the collection of all nonempty proper faces of $K$. Let $\Delta$ be the collection of fans over elements of $K'$. (Notice that, for any element $F \in K'$, the cone we mean here is the cone over $F$, not the cone over $F \times \{1\}$.) Then, for any $F \in K'$, $F = \text{Conv } X$ for some finite set $X$, so that the cone $C$ over $F$ is generated (as a polyhedral cone) by $X$. Moreover, since $K$ is rational, so is $F$, which means that $C$ is a rational cone. Finally, if the $C \cap (-C) \neq \{0\}$ (i.e. if $C$ is not strongly convex), then the same is true of $F$: that is, there exists $x \in F$ such that $-x \in F$. By convexity, this means that the line through $x$ and $-x$ is in $F$. But this line passes through the origin, which is in the interior of $K$. This contradicts the fact that $F$ is contained in the boundary of $K$. So, $C$ must be strongly convex. This proves that the elements of $\Delta$ are strongly convex rational polyhedral cones. Moreover, we have shown that the intersection of faces of $K$ is again a face of $K$, which implies that the intersection of elements of $\Delta$ is again an element of $\Delta$. Moreover, it is clear from the definition that the intersection of faces of $K$ is a face of both the original faces, from which the corresponding statement about cones over $K$ follows. This proves that $\Delta$ is a fan. We can then use this definition of $\Delta$ to associate a fan to any convex polytope.

**Remark.** According to Fulton, we can also generalize the above construction to the case where $K'$ is any collection of rational convex polytopes whose union is the boundary of $K$ such that the intersection of any two polytopes in $K'$ is a polytope in $K'$. However, I do not think we will be needing this generalization.

We might wonder if the fans which come from convex polytopes in the way described above consist of all possible fans. It turns out that this is not the case.

**Example.** Let $C$ be the cube with vertices at $(\pm 1, \pm 1, \pm 1)$ in $\mathbb{Z}^3$, and let $\Delta$ be the fan over the faces of $C$. Let $\Delta'$ be the fan with cones generated by
the same generating sets of cones in $\Delta$ but replacing $(1, 1, 1)$ with $(1, 2, 3)$ in every generating set in which that point appears. One can check that $\Delta'$ does not arise as a fan coming from any polytope. (For more information, see Fulton Section 1.5, p. 25-26.)

The above method of associating a fan to a convex polytope is basically what we want to do. However, it turns out that we can describe the same fan more explicitly by deriving it in a different way. This time, we start with a rational polytope $P$ in the dual space $V^*$. We assume that $P$ is $n$-dimensional, but it need not contain the origin. Then, given any face $Q$ of $P$, we can define

$$\sigma_Q = \{v \in V : \langle u, v \rangle \leq \langle u', v \rangle \text{ for all } u \in Q \text{ and } u' \in P\}.$$ 

It is not immediately obvious that the $\sigma_Q$ are even cones. However, the following proposition tells us that the $\sigma_Q$ are not only cones but are in fact exactly the cones over the faces of the dual polytope $P^0$.

**Proposition 12.** Suppose $P$ is a convex polytope in $V^*$ containing the origin in its interior. Then, for any face $Q$ of $P$, $\sigma_Q$ is the cone over the face $Q^*$ of $P^0$.

**Proof.** Let $C \subset V$ be the cone over $Q^*$. Then, for any $v \in C$, we can write $v = \alpha v'$, where $\alpha \geq 0$ and $v'$ is in $Q^*$. In particular, this means that $v' \in P^0$, so that for any $u' \in P$, we have

$$\langle u', v \rangle = \alpha \langle u', v' \rangle \geq -\alpha.$$ 

On the other hand, since $v'$ is in $Q^*$, we have $\langle u, v' \rangle = -1$ for all $u \in Q$, so that

$$\langle u, v \rangle = \alpha \langle u, v' \rangle = -\alpha.$$ 

This implies that $\langle u, v \rangle \leq \langle u', v \rangle$ for all $u$ in $Q$ and $u'$ in $P$, so that $v \in \sigma_Q$.

Conversely, suppose that $v \in \sigma_Q$. Since 0 is in the interior of $P$ we have for all $u \in Q$ that

$$\langle u, v \rangle \leq \langle 0, v \rangle = 0.$$ 

On the other hand, $\langle u, v \rangle = 0$ if and only if $u$ or $v$ is 0. But 0 is not in $Q$ (since $Q$ lies in the boundary of $P$, while 0 lies in the interior of $P$), so $u \neq 0$. If $v = 0$, then clearly $v \in C$, since $v = 0u$ for any $u \in Q^*$, so we are done. If $v \neq 0$, then by the above, we must have $\langle u, v \rangle < 0$. Suppose $\langle u, v \rangle = -\alpha$, where $\alpha > 0$. Then, $v = \alpha (\frac{1}{\alpha} v)$, and $\langle u, \frac{1}{\alpha} v \rangle = -1$ implies that $\frac{1}{\alpha} v \in Q^*$, so we have $v \in C$. \qed

In light of the above proposition, we define $\Delta_P$ to be the collection of $\sigma_Q$ for all nonempty faces $Q$ of $P$. Then, by the proposition, $\Delta_P$ consists of the cones over the proper faces of $P^0$, which by our above discussion is
a fan. We can then associate to $P$ the toric variety $X(\Delta_P)$, which we will sometimes denote by $X_P$.

Although the proposition requires $P$ to contain the origin in its interior, there is always an affine isomorphism $A$ of $V^*$ which takes $P$ to some convex polytope with the origin in its interior. Moreover, such a transformation preserves less than or greater than between inner products, so $A(\sigma_Q) = \sigma_{A(Q)}$ for all faces $Q$ of $P$. From this it follows that $\Delta_P$ is a fan even when the origin is not in the interior of $P$. (However, it is not necessarily the case that this fan is the same as the fan of cones over the proper faces of $P^0$ in this case.) Moreover, if $A(P) = P'$, then $X_P = X_{P'}$: intuitively, operations like translating, rotating, or stretching $P$ does not really change the structure of the cones in $\Delta_P$, it just applies an isomorphism to the underlying vector space. Since much of the above presentation of polytopes is carried out with the assumption that polytopes contain 0 in their interior (in particular, our definition of the polar of a convex polytope does not work when this is not the case), it is often convenient in practice to use $P^0$ instead of $P$ to compute the toric variety $X_P$.

We end our discussion with a couple examples of toric varieties arising from convex polytopes in the way outlined above.

**Example.** Let $P$ be the standard simplex in $\mathbb{R}^n$: that is, $P = \text{Conv} X$, where $X = \{0, e_1, \ldots, e_n\}$ and the $e_i$ are the standard basis elements of $\mathbb{R}^n$. Then, there is an affine isomorphism which sends $P$ to $P' = \text{Conv}(-X')$, where $X' = \{e_0, e_1, \ldots, e_n\}$ and $e_0 = -\sum_{i=1}^n e_i$. (This is essentially a combination of rotating $P$ about the origin and then stretching $P$ out along the line through the origin and the point $(1, 1, \ldots, 1)$. Since $P'$ contains 0 in its interior, we will use it to compute $X_P$.

One can check that $(P')^0$ is simply $-P' = \text{Conv} X'$ (but in the dual vector space). Now, it is a basic property of $n$-simplices that their faces are simply the convex hull of any subset of their vertices, which for a subset of size $k$ will yield an $(n-k)$-simplex. (Try it out on some low-dimensional examples if you don’t believe this.) In other words, the faces of $-P'$ are precisely the convex hulls of the subsets of $X'$. Since the elements of $\Delta_{P'}$ are cones over the faces of $-P'$, we see that the maximal cones of $\Delta_{P'}$ are $\{\sigma_i\}$, where $\sigma_i$ is the cone generated by $X \setminus \{e_i\}$ (these are precisely the cones over the facets of $-P'$).

To find $A\sigma_i$, we must first compute $\sigma_i'$. To do this, we use the algorithmic way to find the $u_r$ that I described in my notes to Fulton, section 1.2 (p. 8; see also Fulton, p. 11): that is, we take each set of $n-1$ generators of $\sigma_i$ and try to find a vector orthogonal to every vector in this set which is also in $\sigma_i'$. Conveniently, $\sigma_i$ is generated by $n$ vectors, so this is not such a difficult process. In the case where $i = 0$, one can simply see by inspection that $\sigma_0' = \sigma_0$. When $i$ is not zero, $\sigma_i$ is generated by $n$ vectors, so we can throw away any one of them to get a set of $n-1$ generators. If we
throw away $e_0$, we see that $-e_i$ is orthogonal to the remaining generators (which are the $e_j$ such that $j \neq 0, i$) and also satisfies $\langle e_0, -e_i \rangle = 1 \geq 0$, so that $-e_i \in \sigma_i^\vee$. So, $-e_i$ is one of the $u_r$. Throwing away $e_j$ for some $j \neq 0, i$, suppose $u = (u_i)$ is the vector we are looking for. Then, we have $\langle u, e_k \rangle = e_k = 0$ for $k \neq 0, i, j$; $\langle u, e_0 \rangle = -u_i - u_j = 0$; and $\langle u, e_j \rangle = u_j \geq 0$. The most natural choice here is to take $u_j = 1$, so that $u_i = -1$. Then, $u = e_j - e_i$. So, we see that $\sigma_i^\vee$ is generated by $\{-e_i\} \cup \{e_j - e_i\}_{j \neq i \neq 0}$.

Now, for simplicity, we will take $e_i$ to correspond to the element $x_i$ in the $\mathbb{C}$-algebras $A_{\sigma_j}$ for all $i \neq 0$ and all $j$. Then, since the generators for the $\sigma_i^\vee$ that we found are all rational, they generate the semigroups $S_{\sigma}$ as well. So, we can just read off the $\mathbb{C}$-algebras from the above computations:

$$A_{\sigma_0} = \mathbb{C}[x_1, \ldots, x_n],$$

and for all $i \neq 0$,

$$A_{\sigma_i} = \mathbb{C}\left[\frac{x_1}{x_i}, \ldots, \frac{x_{i-1}}{x_i}, \frac{1}{x_i}, \frac{x_{i+1}}{x_i}, \ldots, \frac{x_n}{x_i}\right] = \mathbb{C}[y_1, \ldots, y_n],$$

where $y_j = \frac{x_j}{x_i}$ for all $i \neq j$ and $y_i = \frac{1}{x_i}$.

It is not hard to compute $A_{\sigma_i \cap \sigma_j}$ now. First consider the case where $i = 0$. Now, $\sigma_0 \cap \sigma_j$ is a face of $\sigma_j$, so we can write $\sigma_0 \cap \sigma_j = \sigma_j \cap u^1$ for some $u \in \sigma_j^\vee$. Notice that $\sigma_0 \cap \sigma_j$ is generated by $\{e_k\}_{k \neq 0, j}$. Because of this, we actually already computed the requisite $u$ when we computed the generators of $\sigma_j^\vee$: it is simply $-e_j$. Likewise, we can write $\sigma_0 \cap \sigma_j = \sigma_0 \cap v^1$, where our above computations give us $v = e_j$. Thus, considering $U_{\sigma_0 \cap \sigma_j}$ as a principal open subset of $U_{\sigma_0}$, we have $A_{\sigma_0 \cap \sigma_j} = (A_{\sigma_0})_{x_i} = \mathbb{C}[x_1, \ldots, x_n, x_j^{-1}]$, while considering it as a subset of $U_{\sigma_j}$ gives $A_{\sigma_0 \cap \sigma_j} = (A_{\sigma_j})_{x_i}^{-1} = \mathbb{C}[y_1, \ldots, y_n, y_j^{-1}]$ (recall that here, $y_j = \frac{1}{x_i}$, so that $y_j^{-1} = x_j$).

Our isomorphism on which to glue is then a map

$$\varphi_{0,j} : \mathbb{C}[x_1, \ldots, x_n, x_j^{-1}] \to \mathbb{C}[y_1, \ldots, y_n, y_j^{-1}]$$

defined by sending $x_j \mapsto y_j^{-1} = x_j$ and $x_i \mapsto y_i y_j^{-1} = x_i$ for all $i \neq j$. In a similar fashion, one can check that when $i, j \neq 0$, we have $\sigma_i \cap \sigma_j = \sigma_i \cap (e_j - e_i)^\perp = \sigma_j \cap (e_i - e_j)^\perp$, so that our isomorphism

$$\varphi_{i,j} : (A_{\sigma_i})_{x_i/x_i} = \mathbb{C}[y_1, \ldots, y_n, y_j^{-1}] \to (A_{\sigma_j})_{x_i/x_i} = \mathbb{C}[z_1, \ldots, z_n, z_i^{-1}]$$

(where here we have renamed the $y_k$ in $A_{\sigma_j}$ to $z_k$ to avoid confusion) is defined by sending $y_j = \frac{x_j}{x_i} \mapsto z_i^{-1} = \frac{x_j}{x_i}$, $y_i = \frac{1}{x_i} \mapsto y_i z_i^{-1} = \frac{1}{x_i} = y_j$, and $y_k = \frac{x_k}{x_i} \mapsto z_k z_i^{-1} = \frac{x_k}{x_i}$ for all $k \neq i, j$.

Now that we have specified the gluings that determine the variety structure, we want to compare this structure to common varieties to see what it might be. To this end, we define a map

$$\psi_0 : A_{\sigma_0} \to \mathbb{C}[\frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0}]$$

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by sending $x_i \mapsto \frac{x_i}{x_0}$ for all $i$ and extending linearly in $\mathbb{C}$. One can check that this is a homomorphism of $\mathbb{C}$-algebras and, moreover, that its inverse is given by $\frac{x_i}{x_0} \mapsto x_i$. This is essentially the process of homogenizing our variables: whereas $x_i$ is not a homogeneous function of the $x^j$, $\frac{x_i}{x_0}$ is homogeneous (as a function in $\mathbb{C}(x_0, \ldots, x_n)$). Likewise, for $i \neq 0$, we have a homomorphism of $\mathbb{C}$-algebras $\psi_i : A_{\sigma_i} \to \mathbb{C}[x_0, x_1, \ldots, x_n]/(x_0 x_i)$ by sending $\frac{x_j}{x_i} \mapsto \frac{x_j}{x_i}$ for all $j \neq i$ and sending $\frac{1}{x_i} \mapsto \frac{x_0}{x_i}$, with the natural inverse. Moreover, one can check that, after applying these isomorphisms, the $\varphi_{0,j}$ send $\frac{x_0}{x_0} \mapsto (\frac{x_0}{x_1})^{-1}$, and the $\varphi_{i,j}$ for $i \neq 0$ is the same except that it now sends $y_i = \frac{x_0}{x_i} \mapsto y_j = \frac{x_0}{x_j}$.

Now, recall from Mitchell’s second lecture that the coordinate ring on a principal open subset is the set of regular functions on that subset (i.e. the set of rational functions whose denominator does not vanish on the subset). Moreover, recall (or review either of Mitchell’s lecture notes) that $\mathbb{CP}^n$ is covered by sets of the form $U_i = \{[x_0 : \cdots : x_n] \in \mathbb{CP}^n : x_i \neq 0\}$, with transition functions given by sending $x \mapsto x^{-1}$. By definition of $U_i$, the coordinate ring on $U_i$ is precisely $\mathbb{C}[\frac{x_0}{x_1}, \ldots, \frac{x_n}{x_i}] \cong A_{\sigma_i}$; moreover, the homomorphisms of coordinate rings corresponding to the transition functions are precisely the $\varphi_{i,j}$ as described above. Thus, by comparison, we see that $X_P \cong \mathbb{CP}^n$.

The following example is a lot less involved than the previous one; I’ll leave it as an exercise, in case you want some extra practice.

**Exercise.** Let $P$ be the cube in $\mathbb{R}^3$ with vertices at $\pm e_1^*, \pm e_2^*$, and $\pm e_3^*$. Show that $P^0$ is the octahedron with vertices at $\pm e_1$, $\pm e_2$, and $\pm e_3$, so that $\Delta_P$ is the fan over the faces of this octahedron. Then, show that $X(\Delta_P) \cong \mathbb{CP}^1 \times \mathbb{CP}^1 \times \mathbb{CP}^1$. (Hint: visual the octahedron and pick out its facets geometrically. From this, notes that each of the 8 maximal cones in $\Delta_P$ is one entire octant of $\mathbb{R}^3$. Write $\Delta_P$ as a product of fans, and then apply Proposition 2.)