**Definition 0.1.** Let $R$ be a ring. An ideal $I$ is called **finitely generated** if there are elements $a_1, \ldots, a_n \in R$ such that
\[ I = (a_1, \ldots, a_n) = Ra_1 + \cdots + Ra_n. \]

We say $R$ is **Noetherian** if every ideal is finitely generated.

**Example 0.2.** A field $K$ is Noetherian. The only ideals are $\{0\}$ and $K = (1)$.

**Proposition 0.3.** The ring $\mathbb{Z}$ is Noetherian.

**Proof.** Let $I$ be a nonzero ideal in $\mathbb{Z}$. Let $a$ be a nonzero element in $I$ of smallest norm $|a|$. We claim that $I = (a)$. Indeed, let $b$ be another nonzero element of $I$. By the division algorithm, there are integers $q, r$ satisfying $b = aq + r$ where $|r| < |a|$ or $r = 0$. Since $I$ is an ideal, we find that $r = b - aq \in I$. If $r$ is nonzero, then $r$ has smaller norm than $a$, which is a contradiction. So we must have $r = 0$ and thus $b = aq \in (a)$. \hfill \qed

**Proposition 0.4.** For a field $K$, the ring $K[x]$ is Noetherian.

**Sketch of Proof.** Use the norm given by the degree and use the division algorithm as above.

**Remark 0.5.** The above proofs actually show that the rings $\mathbb{Z}$ and $K[x]$ are principal ideal domains, that is, integral domains in which every ideal is principal. It is clear that a principal ideal domain is Noetherian.

**Definition 0.6.** Let $R$ be a ring. We say $R$ satisfies the **ascending chain condition** if for each chain of ideals
\[ I_0 \subset I_1 \subset \cdots \]
in $R$, there is an $N \geq 0$ such that for each $n \geq N$, we have $I_n = I_N$ (that is, the chain terminates after a finite number of terms).

**Lemma 0.7.** A ring $R$ is Noetherian if and only if it satisfies the ascending chain condition.

**Proof.** Suppose $R$ is Noetherian, and let $I_0 \subset I_1 \subset \cdots$ be an ascending chain of ideals. It follows that the union $I = \cup_{m=0}^\infty I_m$ is an ideal. (Why?) Since $R$ is Noetherian, there are $a_1, \ldots, a_m$ such that $I = (a_1, \ldots, a_n)$. Each $a_i$ is an element of $I_{m(i)}$ for some $m(i)$, so if we set $N = \max\{m(i)\}$, then we see that $I_n = I_N = I$ for each $n \geq N$.

Suppose $R$ is not Noetherian. There is an ideal $I$ of $R$ that is not finitely generated. Pick $a_0 \in I$ and set $I_0 = (a_0)$. Inductively, for each $n > 0$, since $I$ is not finitely generated, we can choose $a_{n+1} \in I \setminus I_n$ and set $I_{n+1} = I_n + (a_{n+1})$. This gives an ascending chain of ideals
\[ I_0 \subset I_1 \subset \cdots \]
which never terminates. Thus $R$ does not satisfy the ascending chain condition.

**Lemma 0.8.** For a surjective ring homomorphism $\pi: R \to S$, we have

$$\pi(a_1, a_2) = (\pi(a_1), \pi(a_2)).$$

**Proof.** Note that $\pi(r_1a_1 + r_2a_2) = \pi(r_1)\pi(a_1) + \pi(r_2)\pi(a_2) \in (\pi(a_1), \pi(a_2))$. On the other hand, if $s_1\pi(a_1) + s_2\pi(a_2) \in (\pi(a_1), \pi(a_2))$, then there are $r_j \in R$ such that $\pi(r_j) = s_j$ and hence

$$s_1\pi(a_1) + s_2\pi(a_2) = \pi(r_1a_1 + r_2a_2).$$

**Proposition 0.9.** If $R$ is a Noetherian ring and $I$ is an ideal, then $R/I$ is Noetherian.

**Proof.** Let $J$ be an ideal in $R/I$. Then $\pi^{-1}(J)$ is an ideal in $R$, so there are a finite number of elements $a_1, \ldots, a_n$ in $R$ such that $\pi^{-1}(J) = (a_1, \ldots, a_n)$. By the previous lemma, we have

$$J = (\pi(a_1), \ldots, \pi(a_n)).$$

**Theorem 0.10** (Hilbert’s Basis Theorem). If $R$ is a Noetherian ring, then $R[x]$ is a Noetherian ring.

**Corollary 0.11.** If $R$ is Noetherian, then the ring $R[x_1, \ldots, x_n]$ is Noetherian.

**Sketch of Proof of Theorem.** Let $I$ be an ideal of $R[x]$. For each $n \geq 0$, let $I_n$ denote the ideal in $R$ of “leading coefficients of elements of $I$ of degree $n$.” This means that $I_n$ consists of all elements $r$ such that there is a polynomial $f \in I$ of the form $f = rx^n + a_{n-1}x^{n-1} + \cdots + a_0$. It is routine to check that $I_n$ is an ideal and that $I_n \subset I_{n+1}$. By the ascending chain condition, there is an $N \geq 0$ such that $I_n = I_N$ for each $n \geq N$.

Now for each $n \leq N$, the ideal $I_n$ is finitely generated, so there are a finite number of generators $\{a^n_i\}_{i=1}^{k_n}$ for $I_n$. By definition, these generators appear as the leading terms of some polynomials $\{f^n_i\}_{i=1}^{k_n}$ of degree $n$ in $R[x]$. We then claim that the set of polynomials

$$\bigcup_{n=0}^{N} \{f^n_i\}_{i=1}^{k_n}$$

generates $I$. Let $I^*$ denote the ideal generated by these polynomials.

Let $g$ be arbitrary in $I$. We prove that $g \in I^*$ by induction on the degree $m$ of $g$. For $m = 0$, the fact that $g \in I$ means that $g \in I_0$ and hence $g$ belongs to $I^*$ since the generators of $I_0$ are also generators of $I^*$.

Suppose now that the claim holds for some $m - 1$. We prove the claim for $m$. There are two cases.
(i) Case 1: Suppose $m \leq N$. In this case, the leading coefficient $b_m$ of $g$ belongs to $I_m$, and hence we can write

$$b_m = \sum_{i=1}^{k_m} c_i a_i^m$$

for some $c_i \in R$. The polynomial

$$h = g - \sum_{i=1}^{k_m} c_i f_i^m$$

then has degree less than $m$. By induction, $h \in I^*$ and since $\sum_{i=1}^{k_m} c_i f_i^m$ belongs to $I^*$, we see that $g$ does as well.

(ii) Case 2: Suppose $m > N$. In this case, the leading coefficient $b_m$ of $g$ belongs to $I_m = I_N$, and hence we can write

$$b_m = \sum_{i=1}^{k_N} c_i a_i^N$$

for some $c_i \in R$. The polynomial

$$h = g - x^{N-m} \sum_{i=1}^{k_N} c_i f_i^N$$

then has degree less than $m$. By induction, we are done again.

**Definition 0.12.** For an ideal $I$ of $\mathbb{C}[x_1, \ldots, x_n]$, define the affine algebraic subset

$$V(I) = \{ x \in \mathbb{C}^n : f(x) = 0 \text{ for each } f \in I \}.$$ 

Because $\mathbb{C}[x_1, \ldots, x_n]$ is Noetherian, the ideal $I$ is finitely generated by some polynomials $f_1, \ldots, f_k$, and hence $V(I)$ is the intersection of finitely many hypersurfaces (Exercise 2.2.2).

**Remark 0.13.**

(a) Recall that ideal $I$ of a ring $R$ is called prime if whenever a product $ab \in I$ of two elements from $R$ belongs to $I$, then either $a \in I$ or $b \in I$.

(b) Recall that the radical of an ideal $I$ in $R$ is the ideal $\sqrt{I}$ defined by

$$\sqrt{I} = \{ a \in R : a^n \in I \text{ for some } n > 0 \},$$

and we say that an ideal $I$ is a radical ideal if $I = \sqrt{I}$.

(c) Recall that any prime ideal is a radical ideal.

(d) It follows easily from (c) that the intersection of any number of prime ideals is radical.

**Example 0.14** (Exercise 2.3.1). We claim that the ideal $I = (xy, yz) \subset \mathbb{C}[x, y, z]$ is radical but not prime. Indeed, it is not prime because the element $xy$ belongs to $I$, but neither $x$ nor $y$ belong to $I$. It is radical.
because it is the intersection of prime ideals \( I = (x) \cap (y, z) \), with both \( (x) \) and \( (y, z) \) being prime as the quotients

\[
\frac{\mathbb{C}[x, y, z]}{(x)} \simeq \mathbb{C}[y, z] \\
\frac{\mathbb{C}[x, y, z]}{(y, z)} \simeq \mathbb{C}[x]
\]

are integral domains.

**Lemma 0.15.** The algebraic subset \( V(I) \) is irreducible if and only if \( I \) is a prime ideal.

*Proof.* Exercise. □

**Definition 0.16.** For a subset \( S \) of \( \mathbb{C}^n \), define the ideal

\[
I(S) = \{ f \in \mathbb{C}[x_1, \ldots, x_n] : f(x) = 0 \text{ for each } x \in S \}
\]

of polynomials vanishing along \( S \).

**Remark 0.17.** We have defined two operations

\[
V : \{ \text{ideals of } \mathbb{C}[x_1, \ldots, x_n] \} \to \{ \text{algebraic subsets of } \mathbb{C}^n \}
\]

and

\[
I : \{ \text{subsets of } \mathbb{C}^n \} \to \{ \text{ideals of } \mathbb{C}[x_1, \ldots, x_n] \}.
\]

The next theorem, called the Nullstellensatz, states that these two operations are inverses to one another, when suitable conditions are placed on the domains and codomains.

**Theorem 0.18** (Nullstellensatz). The operators \( V \) and \( I \) induce a bijective correspondence

\[
\{ \text{radical ideals of } \mathbb{C}[x_1, \ldots, x_n] \} \leftrightarrow \{ \text{algebraic subsets of } \mathbb{C}^n \}.
\]

Moreover, the correspondence restricts to a correspondence

\[
\{ \text{prime ideals of } \mathbb{C}[x_1, \ldots, x_n] \} \leftrightarrow \{ \text{affine subvarieties of } \mathbb{C}^n \}.
\]

**Definition 0.19.** A ring \( R \) is called **reduced** if the zero ideal is radical.

**Lemma 0.20.** An integral domain is reduced.

*Proof.* If \( R \) is an integral domain, then the zero ideal is prime, and hence radical. □
Example 0.21. The ring \( \mathbb{C}[x,y]/(xy) \) is reduced but not an integral domain. This is because the ideal \((xy)\) is radical, but not prime.

Definition 0.22. For an affine algebraic set \( X \subset \mathbb{C}^n \), define the **coordinate ring** \( R(X) \) to be the quotient \( R(X) = \mathbb{C}[x_1, \ldots, x_n]/I(X) \). Since the ideal \( I(X) \) is radical the quotient \( R(X) \) is reduced, and moreover, generated by the images of \( x_1, \ldots, x_n \) (Exercise 2.4.1).

Lemma 0.23. If \( \varphi : R \to S \) is a ring homomorphism and \( I \subset R \) is an ideal satisfying \( I \subset \ker(\varphi) \), then there is a unique ring homomorphism \( \tilde{\varphi} : R/I \to S \) making the following diagram commute:

\[
\begin{array}{ccc}
R & \xrightarrow{\varphi} & S \\
\downarrow{\pi} & & \downarrow{\pi_S} \\
R/I & \xrightarrow{\tilde{\varphi}} & S/J
\end{array}
\]

Proof. Define \( \tilde{\varphi} \) by the rule \( \tilde{\varphi}(r + I) = \varphi(r) \). This is well-defined because if \( i \in I \), then \( \varphi(r + i) = \varphi(r) + 0 = \varphi(r) \) since \( I \subset \ker \varphi \). Moreover, it is a ring homomorphism because \( \varphi \) is.

Lemma 0.24. If \( \varphi : R \to S \) is a ring homomorphism and \( I \subset R \) and \( J \subset S \) are ideals such that \( \varphi(I) \subset J \), then there is a unique ring homomorphism \( \tilde{\varphi} : R/I \to S/J \) such that the following diagram commutes:

\[
\begin{array}{ccc}
R & \xrightarrow{\varphi} & S \\
\downarrow{\pi_R} & & \downarrow{\pi_S} \\
R/I & \xrightarrow{\tilde{\varphi}} & S/J
\end{array}
\]

Proof. Apply the lemma to \( \pi_S \circ \varphi \).

Definition 0.25. Let \( F : \mathbb{C}^m \to \mathbb{C}^n \) be a polynomial map. Define a map \( F^\sharp : \mathbb{C}[y_1, \ldots, y_n] \to \mathbb{C}[x_1, \ldots, x_m] \) by rule

\[
F^\sharp \sigma(x_1, \ldots, x_m) = \sigma(F(x_1, \ldots, x_m)) \quad \text{for} \quad \sigma \in \mathbb{C}[y_1, \ldots, y_n].
\]

Because \( F \) is a polynomial map, it follows that \( F^\sharp \) is a ring homomorphism.

Lemma 0.26. If \( F : \mathbb{C}^m \to \mathbb{C}^n \) is a polynomial map and \( X \subset \mathbb{C}^m \) and \( Y \subset \mathbb{C}^n \) are varieties such that \( F(X) \subset Y \), then \( F^\sharp(I(Y)) \subset I(X) \).

Definition 0.27. Let \( X \subset \mathbb{C}^m \) and \( Y \subset \mathbb{C}^n \) be two algebraic subsets, and let \( f : X \to Y \) be a morphism. Remember that this means that \( f \) is the restriction of a polynomial map \( F : \mathbb{C}^m \to \mathbb{C}^n \). Because \( F^\sharp(I(Y)) \subset I(X) \), there is a unique ring homomorphism \( f^\sharp : R(Y) \to R(X) \), called the **pullback map**, making the following diagram commute:

\[
\begin{array}{ccc}
\mathbb{C}[y_1, \ldots, y_n] & \xrightarrow{F^\sharp} & \mathbb{C}[x_1, \ldots, x_m] \\
\downarrow & & \downarrow \\
R(Y) & \xrightarrow{f^\sharp} & R(X)
\end{array}
\]
Thus \( f^\sharp \) is the ring homomorphism satisfying the property that

\[
f^\sharp \sigma(y_1, \ldots, y_n) = \sigma(f(y_1, \ldots, y_n)) \quad \text{for } \sigma \in R(Y)
\]

where we understand \( \sigma(f(y_1, \ldots, y_n)) \) to denote an element of \( R(X) \).

**Example 0.28.** Let \( X = V(y - x^2) \subset \mathbb{C}^2 \) and \( Y = \mathbb{C}^1 \). Then \( X \) is isomorphic to \( \mathbb{C}^1 \) via the morphism

\[
f : X \to \mathbb{C}^1
\]

\[(x, y) \mapsto x.\]

Note that \( f \) is induced via the polynomial map \( F : \mathbb{C}^2 \to \mathbb{C}^1 \) determined by \((x, y) \mapsto x\). Remember that the inverse of \( f \) is \( g : \mathbb{C}^1 \to X \) given by \( t \mapsto (t, t^2) \), which is determined by the global map \( G : \mathbb{C}^1 \to \mathbb{C}^2 \) given by \( t \mapsto (t, t^2) \).

The pullback map \( F^\sharp : \mathbb{C}[z] \to \mathbb{C}[x, y] \) is determined by the assignment \( z \mapsto x \). The composition \( f^\sharp = \pi \circ F^\sharp : \mathbb{C}[z] \to R(X) \) is then a well-defined ring homomorphism.

The pullback map \( G^\sharp : \mathbb{C}[x, y] \to \mathbb{C}[z] \) is determined by the assignments

\[
x \mapsto z
\]

\[
y \mapsto z^2.
\]

Because \( G^\sharp(I(X)) = G^\sharp(y - x^2) = 0 \), we see that \( G^\sharp \) induces a well-defined ring homomorphism \( g^\sharp : R(X) \to \mathbb{C}[z] \).

The ring homomorphisms \( f^\sharp : \mathbb{C}[z] \to R(X) \) and \( g^\sharp : R(X) \to \mathbb{C}[z] \) are inverses to one another, and hence define ring isomorphisms. We conclude that

\[
\mathbb{C}[x, y]/(y - x^2) \simeq R(X) \simeq \mathbb{C}[z].
\]

**Remark 0.29.** Note that we have constructed two operations

\[
R : \{ \text{algebraic subsets of } \mathbb{C}^n \} \to \{ \text{finitely generated reduced } \mathbb{C}\text{-algebras} \}
\]

\[
\sharp : \{ \text{morphisms } f : X \to Y \} \to \{ \text{morphisms } f^\sharp : R(Y) \to R(X) \}
\]

In some sense, these two operations define an “equivalence of categories,” which Augusto will discuss more next time.

**Exercise 0.30.** Show that for morphisms \( f : X \to Y \) and \( g : Y \to Z \) of affine algebraic subsets we have \((g \circ f)^\sharp = f^\sharp \circ g^\sharp\).

**Exercise 0.31.** Show that for an affine algebraic subset \( X \), we have \( \text{id}^\sharp_X = \text{id}_{R(X)} \).
Exercise 0.32. Show that if $f : X \to Y$ is an isomorphism of affine algebraic subsets, then $f^\dagger : X \to Y$ is an isomorphism of rings (or $\mathbb{C}$-algebras).