1. I feel like I didn’t explain my response to Andrew’s question properly, so here is an exercise which fills in the details better. Recall that a map \( f : X \to Y \) is open if and only if \( f(U) \) is open for each open \( U \subset X \). Andrew asked whether there were examples of projection maps \( \pi : X \to X/\sim \) which are not open. The answer is yes, and here is one line of reasoning that I tried to outline on the board.

(i) For a topological space \( X \), a set \( Y \), and a surjective mapping \( f : X \to Y \), show that the set \( \{ V \subset Y : f^{-1}(V) \text{ is open in } X \} \) is a topology on \( Y \) called the quotient topology.

(ii) When \( Y \) is equipped with the quotient topology, show that \( f \) is continuous.

(iii) Define an equivalence relation \( \sim \) on \( X \) by \( x_1 \sim x_2 \) if and only if \( f(x_1) = f(x_2) \), and let \( X/\sim \) denote the quotient space. Show that \( f \) induces a well-defined map \( g : X/\sim \to Y \) described by \( g([x]) = f(x) \).

(iv) Show that \( g \) is continuous.

(v) Show that \( g \) is bijective and hence admits an inverse \( g^{-1} : Y \to X/\sim \).

(vi) Show that the inverse of \( g \) is continuous, and hence \( g \) is a homeomorphism.

(vii) Show that \( f \) is open if and only if \( \pi \) is.

(viii) As sets, let \( X = Y = [0,1] \). Equip \( X \) with the standard topology. Let \( f : X \to Y \) be defined by

\[
 f(x) = \begin{cases} 
 0 & 0 \leq x \leq 1/2 \\
 2x - 1 & 1/2 \leq x \leq 1. 
\end{cases}
\]

Equip \( Y \) with the quotient topology induced by \( f \). Show that \( f \) is continuous but not open. Conclude that the corresponding projection map \( \pi : X \to X/\sim \) is not open.

2. Recall that an abelian semigroup is a set \( S \) together with a binary operation \( * : S \times S \to S \) such that \( s_1 * s_2 = s_2 * s_1 \) for each \( s_1, s_2 \in S \). We say that \( S \) has a unit if there is an element \( e \in S \) such that \( e * s = s \) for each \( s \in S \). Theorem 4.6.2 says that \( \mathbb{K} \) is an abelian semigroup with unit given by the unknot.

(i) If \( S \) has a unit, show that it is unique. (That is, if \( e_1, e_2 \) are two units for \( S \), show that \( e_1 = e_2 \).)

(ii) We say that an element \( s \in S \) divides another element \( r \in S \), if there is an element \( t \in S \) such that \( s * t = r \). Show that the unknot divides every knot.

(iii) We say that an element \( s \) is prime in \( S \) if whenever \( s \) divides a product \( a * b \), either \( s \) divides \( a \) or \( s \) divides \( b \). Show that if \( K_P \) is a prime knot, then \( K_P \) is a prime element of \( \mathbb{K} \).

(iv) We say that a non-unit \( s \in S \) is irreducible if whenever \( s = s_1 * s_2 \) for some \( s_1, s_2 \in S \), either \( s_1 = e \) or \( s_2 = e \). Show that every prime number is irreducible in \((\mathbb{N}_{>0}, \cdot)\).

(v) Show that every prime knot is irreducible in \( \mathbb{K} \).

(vi) We say that an abelian semigroup with unit \( S \) has unique factorization if for each element \( s \in S \) there are irreducible elements \( s_1, \ldots, s_n \in S \) such that

\[
 s = e * s_1 * \cdots * s_n
\]

and this representation is unique in the sense that if

\[
 s = e * t_1 \cdots * t_m
\]

for some \( t_1, \ldots, t_m \in S \), then \( m = n \) and there is a bijection \( \phi : \{1, \ldots, m\} \to \{1, \ldots, n\} \) such that \( t_{\phi(i)} = s_i \) for each \( i \). Why does \( \mathbb{K} \) have unique factorization?
3. This exercise supplements the proof of Lemma 4.7.1 in Cromwell. Let \( v = (v_1, v_2, v_3) \) be a vector in \( \mathbb{R}^3 \) such that \( v_3 \neq 0 \). If \( H_+ = \{ x_3 > 0 \} \subset \mathbb{R}^3 \) and \( H_- = \{ x_3 < 0 \} \subset \mathbb{R}^3 \), then either \( v \in H_+ \) or \( v \in H_- \).

   (i) If \( v \in H_\pm \), show that there is a unique linear transformation \( L : \mathbb{R}^3 \to \mathbb{R}^3 \) satisfying \( L(e_1) = e_1 \), \( L(e_2) = e_2 \), and \( L(v) = \pm e_3 \).

   (ii) Conclude that \( L \) is the identity on the \( x_1x_2 \)-plane, and sends \( v \) to a vector perpendicular to this plane.

   (iii) Show that \( L \) is orientation preserving.

   (iv) Show that there is an isotopy from the identity map to \( L \).

4. Cromwell 4.11.2

5. Cromwell 4.11.7

6. Cromwell 4.11.8

7. Cromwell 4.11.9