Note! For sets $X, Y$, I write $X \subset Y$ to mean that $X$ is a subset of $Y$. If I want to indicate that $X$ is a proper subset of $Y$, then I write $X \subsetneq Y$.

1. For a subset $S$ of a topological space $X$, we say that a point $x \in S$ is an interior point of $S$ if there is an open subset $U$ of $X$ such that $x \in U \subset S$. We let $\text{Int}(S)$ denote the set of interior points of $S$.

   (i) Show that $\text{Int}(S) = \bigcup_{U \text{ open}} U$ and deduce that $\text{Int}(S)$ is open.

   (ii) Proof or counterexample: $S$ is open if and only if $S = \text{Int}(S)$.

   (iii) Proof or counterexample: if $S_1 \subset S_2$, then $\text{Int}(S_1) \subset \text{Int}(S_2)$.

   (iv) Proof or counterexample: $\text{Int}(S_1 \cap S_2) = \text{Int}(S_1) \cap \text{Int}(S_2)$.

   (v) Proof or counterexample: $\text{Int}(S_1 \cup S_2) = \text{Int}(S_1) \cup \text{Int}(S_2)$.

2. For a subset $S$ of a topological space $X$, we let $\text{Cl}(S)$ denote the closure of $S$:

   \[ \text{Cl}(S) = \bigcap_{C \text{ closed}} C. \]

   (i) Show that the intersection of closed sets is closed and deduce that $\text{Cl}(S)$ is the “smallest closed set containing $S$.”

   (ii) Proof or counterexample: $S$ is closed if and only if $\text{Cl}(S) = S$.

   (iii) Proof or counterexample: if $S_1 \subset S_2$, then $\text{Cl}(S_1) \subset \text{Cl}(S_2)$.

   (iv) Proof or counterexample: $\text{Cl}(\text{Cl}(S)) = \text{Cl}(S)$.

   (v) Proof or counterexample: $\text{Cl}(S_1 \cap S_2) \subset \text{Cl}(S_1) \cap \text{Cl}(S_2)$.

   (vi) Proof or counterexample: $\text{Cl}(S_1 \cap S_2) = \text{Cl}(S_1) \cap \text{Cl}(S_2)$.

   (vii) Proof or counterexample: $\text{Cl}(S_1 \cup S_2) = \text{Cl}(S_1) \cup \text{Cl}(S_2)$.

   (viii) Proof or counterexample: $\text{Cl}(S) = X \setminus (\text{Int}(X \setminus S))$.

3. For a subset $S$ of a topological space $X$, we say that $x \in X$ is a limit point of $S$ if each open neighborhood $U$ of $x$ intersects $S$ in at least one point other than $x$ itself.

   (i) Show that $S$ is closed if and only if it contains all its limit points.

   (ii) If $L(S)$ denotes the set of limit points of $S$, deduce that $\text{Cl}(S) = S \cup L(S)$.

   (iii) Find $L(B_1(0))$ where $B_1(0) = \{x \in \mathbb{R}^n : |x| < 1\}$.

   (iv) Find $L((a,b))$ and $L([a,b])$.

4. Define the boundary of $S \subset X$ to be $\partial S = \text{Cl}(S) \setminus \text{Int}(S)$. 

1
(i) Show that \( \partial(S_1 \cap S_2) \subset (\partial S_1 \cap \text{Cl}(S_2)) \cup (\text{Cl}(S_1) \cap \partial S_2) \). But show that the reverse inclusion is in general not true.

(ii) Show that \( \partial \) satisfies the Leibniz rule \( \partial(S_1 \cap S_2) = (\partial S_1 \cap S_2) \cup (S_1 \cap \partial S_2) \) if both \( S_1 \) and \( S_2 \) are closed.

(iii) Proof or counterexample: \( \partial S \) is closed.

(iv) Find \( \partial(B_1(0)) \) where \( B_1(0) = \{ x \in \mathbb{R}^n : |x| < 1 \} \).

(v) Find \( \partial(S^n) \) where \( S^n = \{ x \in \mathbb{R}^{n+1} : |x| = 1 \} \).

(vi) Find \( \partial(D) \) where \( D = \{ x \in \mathbb{R}^n : |x| \leq 1 \} \).

(vii) Find \( \partial(\mathbb{Q}) \) where \( \mathbb{Q} \subset \mathbb{R} \) denotes the subset of rational numbers.

(viii) Find \( \partial(\mathbb{Q}) \) where \( \mathbb{Q} \subset \mathbb{R} \) denotes the subset of rational numbers.

(ix) Proof or counterexample: \( \partial \partial S = \partial S \).

(x) Proof or counterexample: \( \partial \partial \partial S = \partial \partial S \).

5. Let \( X \) be a topological space. Consider the following conditions on \( X \)

(a) each point \( x \in X \) admits an open neighborhood \( U \ni x \) and a continuous map \( \phi : U \to \mathbb{R}^n \) taking \( U \) homeomorphically onto the open unit ball \( B_1(0) \subset \mathbb{R}^n \)

(b) each point \( x \in X \) admits an open neighborhood \( U \ni x \) and a continuous map \( \phi : U \to \mathbb{R}^n \) taking \( U \) homeomorphically onto an open subset \( \phi(U) \subset \mathbb{R}^n \)

(c) each point \( x \in X \) admits an open neighborhood \( U \ni x \) and a continuous map \( \phi : U \to \mathbb{R}^n_{\geq 0} = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_1 \geq 0 \} \) taking \( U \) homeomorphically onto an open subset \( \phi(U) \subset \mathbb{R}^n_{\geq 0} \).

(i) Construct a homeomorphism from \( \mathbb{R}^n_{\geq 0} = \{(x_1, \ldots, x_n) : x_1 > 0 \} \) onto \( \mathbb{R}^n \).

(ii) Show that \( (a) \implies (b) \implies (c) \).

(iii) Show that \( (b) \implies (a) \) (and hence \( (a) \iff (b) \))

(iv) Proof or counterexample: \( (c) \implies (b) \).

6. For \( m \geq n \), and for a map \( \sigma : \{1, 2, \ldots, m\} \to \{1, 2, \ldots, n\} \), let \( f_\sigma : \mathbb{R}^m \to \mathbb{R}^m \) be defined by

\[
f_\sigma(x_1, \ldots, x_m) = (x_{\sigma(1)}, \ldots, x_{\sigma(m)}).
\]

Find necessary and sufficient conditions on \( m, n \), and \( \sigma \) so that \( f_\sigma \) is a homeomorphism.

7. Let \( \{U_\alpha\} \) be an open cover of a topological space \( X \).

(i) Proof or counterexample: a subset \( V \) is open in \( X \) if and only if each \( V \cap U_\alpha \) is open in \( U_\alpha \).

(ii) Proof or counterexample: a subset \( C \) is closed in \( X \) if and only if each \( C \cap U_\alpha \) is closed in \( U_\alpha \).

8. For a space \( X \) satisfying condition (c) above, let \( \delta X \) denote those points \( x \in X \) which admit an open neighborhood \( U \ni x \) and a continuous map \( \phi : U \to \mathbb{R}^n_{\geq 0} \) taking \( U \) homeomorphically onto \( \phi(U) \) such that \( \phi(x) \in \{ x_1 = 0 \} \subset \mathbb{R}^n \). Proof or counterexample: If \( X \subset \mathbb{R}^n \), then \( \partial X = \delta X \).

9. Cromwell 3.10.6

10. Cromwell 3.10.9

11. Cromwell 3.10.14 (Note that a **chessboard colouring** is defined in 3.10.13.)

12. Cromwell 4.11.2