1 Graphs

In this section we will define graphs in the combinatorial sense.

1.1 Definitions and Facts

Definition A graph is a finite set of vertices $V$, and a finite set of edges $E$. We represent each edge as an unordered pair $[v_i, v_j] \in V \times V$ of the vertices in which they connect. The vertices $v_i$ and $v_j$ are the endpoints of the edge.

E.g. An example of a graph is the following:

![Graph Example](image)

Definition An edge with the same vertex at each end is called a loop. When an endpoint shares two or more edges we have a multiple edge. A graph without loops or multiple edges is called simple.

E.g. An example of a multiple edge is $v_3$ in Figure 1.1. An example of a loop is the following:

![Loop Example](image)
**Definition** The *valence* (or *degree*) of a vertex \( v \) is the number of times the vertex appears in the list of unordered pairs \( E \) (or the number of edges with endpoint \( v \), counting loops twice).

**Definition** A *path* in a graph is a sequence of edges that can be written as \([v_0, v_1], [v_1, v_2], [v_2, v_3], \ldots, [v_{n-1}, v_n]\). A path is *simple* if all the \( v_j \) are distinct. A *circuit* in a graph is a path in which \( v_0 = v_n \). A circuit is *simple* if all the \( v_j \) are distinct except for \( v_0 = v_n \).

**E.g.** An example of a path would be \([v_0, v_1], [v_1, v_2], [v_2, v_3]\) in Figure 1.1.

**Definition** A graph is *connected* if there is a path between any two vertices. A *cut vertex* (or *articulation point*) is a vertex whose removal disconnects the graph. An *isthmus* is an edge whose removal disconnects the graph.

**E.g.** The following is an example of a connected graph with a cut vertex:

An isthmus could be illustrated by two vertices joined with one edge.

**Definition** A *tree* is a connected graph with no circuits. The vertices of a tree can be referred to as *nodes*. Vertices in a tree with valence = 1 are called *terminal nodes*, and their respective edges are called *leaves*.

**E.g.** A tree could be the following:
Denote a graph $G$ with the set of vertices $V$ and edges $E$ as $G = (V, E)$.

**Definition** A graph $G' = (V', E')$ is a *subgraph* of a graph $G = (V, E)$ if $V' \subseteq V$ and $E' \subseteq E$.

**Fact** A tree has at least two terminal nodes.

**Fact** A tree with $n$ nodes has $n - 1$ edges.

*Proof.* By induction.

*Base Case:* It is clear that a tree with 1 node has no edges. *Inductive Case:* Consider a tree $T_{n+1}$ with $n + 1$ nodes. By the previous fact, there is a terminal node, so consider the sub-tree $T_n$ created by removing the terminal node and its corresponding leaf. By our inductive assumption $T_n$ has $n$ nodes and $n - 1$ edges. By adding the node and edge that we removed from $T_{n+1}$, we deduce that $T_{n+1}$ has $n$ nodes and $n - 1$ edges.

In knot theory, we will mostly deal with graphs embedded into planes. We call a graph embedded into a plane a *plane* graph. A graph embedded into a plane will create regions bounded by the edges of the graph, and an unbounded region.

**Definition** The bounded region of a plane graph is called a *face*. A face $F$ is *star shaped* if it contains some point $x$ such that a straight line from $x$ to any point on $\partial F$ lies entirely within $F$. The point $x$ is called a *star point*.

This definition allows us to derive the following theorem about plane graphs.

**Fact (Fáry’s Theorem)** A simple plane graph can be embedded in $\mathbb{R}^2$ so that each edge is a straight line.
Note The proof of this given by Cromwell is merely a sketch, and is by induction on the vertices of a planar graph embedded in $\mathbb{R}^2$, using the concept of star points.

1.2 The Combinatorial Approach

By thinking of knots as piecewise linear we can now accomplish two things:

1. We remove the potential for wild behavior in knots.
2. We can use combinatorial methods to study them.

The following is an example of what thinking of knots as piecewise linear can allow us to accomplish.

**Definition** Let $L$ be a polygonal link embedded in $\mathbb{R}^2$, and let $\Delta$ be a triangle such that:

- $L$ does not meet the interior of $\Delta$
- $L$ meets one or two sides of $\partial \Delta$
- The vertices of $L$ in $L \cap \Delta$ are also vertices of $\Delta$
- The vertices of $\Delta$ in $L \cap \Delta$ are also vertices of $L$

We then define a $\Delta$-move on $L$ as a move that replaces $L$ with $(L - (L \cap \Delta)) \cup (\partial \Delta - L)$.

**E.g.**

This brings us to the next definition:
**Definition** Two polygonal links $L_1$ and $L_2$ are said to be *combinatorically equivalent* if there is a finite sequence of $\Delta$-moves that transforms $L_1$ into $L_2$.

**Note** It is clear that a $\Delta$-move can be achieved by an ambient isotopy, and therefore we deduce that any links that are combinatorially equivalent are also ambient isotopic. The converse is also true, which leads us to the last important fact of this section:

**Fact** If two polygonal links are equivalent under ambient isotopy then they are also combinatorially equivalent.

Source: *Knots and Links* by Peter Cromwell