Factorizing Links & Prime Satellites

4.3 & 4.4

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*The figures on pages 82 and 83 in Cromwell may be useful to accompany many of these definitions.

4.3 Factorizing Links

**Definition: Product Link**

Let $S$ be a 2-sphere which meets a link, $L$, transversely at exactly two points. Let $\alpha \subset S$ be any arc in $S$ that connects the points of $L \cap S$. Let $U_1$ and $U_2$ be the two components of $\mathbb{R}^3 \setminus S$. Define two new links:

$$L_i = (L \cap U_i) \cup \alpha \quad \text{for } i = 1 \text{ and } 2.$$

$L$ is said to be a **product link** with factors $L_1$ and $L_2$. $S$ is called a **factorizing sphere**. Symbolically, we write:

$$L = L_1 \# L_2$$

**Note:** Each factor of a product link is also a companion of it: the companion torus is called a **swallow-follow torus**.

**Note:** The trivial knot is a factor of every link.

**Theorem:** The trivial knot has no non-trivial factors.

**Proof:** The factors of a knot are also its companions. Since we proved last time that the trivial knot has no non-trivial companions, it must likewise have no non-trivial factors. QED.

**Definition: Proper Factor**

A factor of a link is a **proper factor** if it is not the trivial knot and it is not equal to the link, itself.

- A link with proper factors is called **composite**.
- A link with no proper factors is called **locally trivial**.
Definition: Prime Link
A link is a prime link if it is non-trivial, non-split, and locally trivial.

Definition: Simple Link
A non-trivial link is a simple link if it is prime and has no proper companions

Note: Any knot with no proper companions is prime. However, for links, this is not the case.

4.4 Prime Satellites

Definition: Wrapping/Winding Number
Let $W$ be an unkotted solid torus containing a pattern link, $P$. One measure of the complexity of a pattern is the minimum number of intersections $P$ makes with any meridional disc of $W$. This can be considered either absolutely or algebraically (in which case $P$ is oriented and the sign of the intersection is taken into account). The minimum absolute intersection number is called the wrapping number of the pattern. The algebraic intersection number is called the winding number of the pattern.

Note: for the algebraic intersection number, we do not need to take a minimum since the algebraic intersection number is the same for all meridional discs in general positions with respect to $P$.

Classifications by Wrapping Number:
Let $\omega_P$ denote the wrapping number of a pattern, $P$

$\omega_P = 0 \implies P$ sits inside a ball in $W$ and the satellite is unchanged (i.e. $S = P$)
$\omega_P = 1 \implies$ the companion torus is a swallow-follow torus and the satellite construction produces a product link
$\omega_P \geq 2 \implies P \subset W$ is called a proper pattern

$\implies$ if a satellite has a proper pattern and the companion knot is non-trivial, then $S$ is called a proper satellite

Theorem: A proper satellite is prime if its pattern is a prime knot or the trivial knot.

Proof: Let $K$ be a satellite knot with companion solid torus, $V$, and pattern, $P \subset W$. Since $K$ is a proper satellite, every meridional disc of $V$ meets $K$ in at least two points. Let $S$ be a factorizing sphere which decomposes $K$ as a product. We assume that $K$, $S$, and $\partial V$ are in general position.
We now consider the intersection of two surfaces: the sphere, $S$, and the torus, $\partial V$. Since there are both closed surfaces, the intersection must be a (possibly empty) set of loops.

Suppose that $S \cap \partial V \neq \emptyset$ and choose a loop, $\lambda \in S \cap \partial V$, which bounds a disc, $\Delta$ in $S$, s.t. $\Delta \cap \partial V = \partial \Delta = \lambda$. Since there are at least two innermost discs and $K \cap S$ contains only two points, it is possible to choose $\Delta$ so that it meets $K$ in at most one point.

If $\lambda$ bounds a disc in $\partial V$, then we can perform surgery to remove the intersection (as we did in a proof yesterday). Therefore, we can assume that no loops in $S \cap \partial V$ bound discs in $\partial V$.

This leaves us with two cases to consider:

1. Suppose $\Delta$ is outside $V$ and $\lambda$ is a longitude of $\partial V$. Any longitude of $\partial V$ is isotopic to the companion knot which, since it is spanned by the disc, $\Delta$, must be the trivial knot. This contradicts the fact that $K$ is a proper satellite.

2. Suppose $\Delta$ is inside $V$ and $\lambda$ is a meridian of $\partial V$. Since every meridional disc of $V$ meets $K$ in at least two points, $\Delta$ must contain both points of $K \cap S$, contrary to assumption.

Both cases lead to a contradiction so there is no such loop, $\lambda$, and $S \cap \partial V = \emptyset$.

Therefore, $S$ lies inside $V$ and its preimage, $h^{-1}(S)$ in $W$ decomposes the pattern, $P$, as a product. This is also a contradiction and the factorizing sphere cannot exist. Therefore, the pattern must either be prime or trivial.

**Corollary:** Doubles of non-trivial knots are prime.

**Proof:** The pattern is the trivial knot.

**Corollary:** Cable knots are prime.

**Proof:** Torus knots are prime.