# Composition and Differentiation of Species

Monday, October 3, 2022 2:13 AM

**Last Time.**

Addition and Multiplication of Species.
We defined $F+G$ and $F\cdot G$ for $F, G$ species solely based on the enumeration of their structures:
- The number of $(F+G)$-structures on $n$ elements is $|(F+G)[n]| = |F[n]| + |G[n]|$.
- The number of $(F\cdot G)$-structures on $n$ elements is $|(F\cdot G)[n]| = \sum_{i+j=n} \frac{n!}{i!\,j!} |F[i]|\,|G[j]|$.

**Today.**

We want to define $F\circ G$ and $F'$ solely based on the enumeration of their structures:
- $|(F\circ G)[n]| = \sum_{j=0}^{n} \sum_{\substack{n_1+\cdots+n_j=n \\ n_j>0}} \frac{1}{j!} \binom{n}{n_1, \dots, n_j} |F[j]| \prod_{i=1}^{j} |G[n_i]|$.
- $|F'[n]| = |F[n+1]|$.

## Composition of Species.

**Motivation.**

Consider an endofunction (endomorphism) $\varphi \in \mathrm{End}[U]$ of a set $U$.
Two points may be distinguished as recurrent points ($\varphi^k(x)=x$ for some $k$) or nonrecurrent points.
Then, we may naturally identify $\varphi$ with a permutation of disjoint rooted trees.

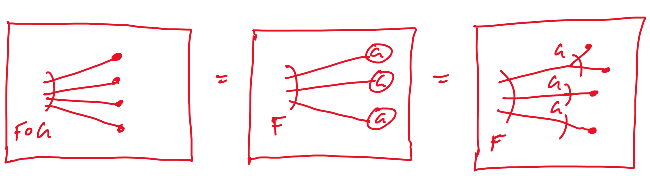
$$\varphi \to \mathrm{End}, \quad \text{permutation} \to S, \quad \text{of} \to \circ, \quad \text{rooted trees} \to A.$$

Thus, $\mathrm{End} = S\circ A$, as species.

What is $\circ$?

**Definition.**

Let $F$ and $G$ be two species such that $G[\phi]=\phi$. Then, $F\circ G = F(G)$ is defined as such:
An $(F\circ G)$-structure on $U$ is a triplet $s=(\pi, \varphi, \gamma)$ such that
(i) $\pi$ is a partition of $U$
(ii) $\varphi$ is an $F$-structure on the set of classes of $\pi$ ($\{[\pi]\}$).
(iii) $\gamma = (\gamma_p)_{p\in\pi}$, where for each $p\in\pi$, $\gamma_p$ is a $G$-structure on $p$.
Then, $(F\circ G)[U] = \sum_{\pi\in \mathrm{Par}[U]} F[\pi] \times \prod_{p\in\pi} G[p]$.




For a bijection $\sigma: U\to V$, for any $(F\circ G)$-structure $s=(\pi, \varphi, (\gamma_p)_{p\in\pi})$,
$(F\circ G)[\sigma](s) = (\bar{\pi}, \bar{\varphi}, (\bar{\gamma})_{\bar{p}\in\bar{\pi}})$,
where
(i) $\bar{\pi}$ is a partition of $V$ obtained by transport of $\pi$ along $\sigma$.
(ii) For each $\bar{p} = \sigma(p) \in \bar{\pi}$, $\bar{\gamma}_{\bar{p}}$ is the $G$-transport of $\gamma_p$ along $\sigma$.
(iii) $\bar{\varphi}$ is the $F$-transport of $\varphi$ along $\bar{\sigma}$ (induced on $\pi$ by $\sigma$).
We say $(F\circ G)$-structures are $F$-assemblies of $G$-structures.

**Theorem.**

(i) $(F\circ G)(x) = F(G(x))$ ← E.G.F is the easiest to work with.

(ii) $(\widetilde{F\circ G})(x) = Z_F(\tilde{G}(x), \tilde{G}(x^2), \dots)$

(iii) $Z_{F\circ G}(x_1, x_2, x_3, \dots) = Z_F(Z_G(x_1, x_2, \dots), Z_G(x_2, x_4, \dots))$ ← Plethystic substitution.

Definition.

$f = f(x_1, x_2, \dots)$, $g = g(x_1, x_2, \dots)$ power series. The plethystic substitution $f\circ g$ is $(f\circ g)(x_1, x_2, \dots) = f(g_1, g_2, \dots)$, where $g_i = g(x_i, x_{2i}, x_{3i}, \dots)$.

Example.

$End = S\circ A \Longrightarrow$

(i) $\sum_{n\ge 0} n^n \frac{x^n}{n!} = End(x) = (S\circ A)(x) = S(A(x)) = \sum n! \frac{(A(x))^n}{n!} = \frac{1}{1-A(x)}$.

(ii) $\widetilde{End}(x) = (\widetilde{S\circ A})(x) = Z_S(\tilde{A}(x), \tilde{A}(x^2), \dots) = \frac{1}{(1-\tilde{A}(x))(1-\tilde{A}(x^2))\cdots}$

(iii) $Z_{End} = (Z_S\circ Z_A)(x_1, \dots, x_i, \dots) = \frac{1}{(1-x_1\circ Z_A)(1-x_2\circ Z_A)\cdots}$.

Species of Partitions Par.

Every partition may be identified with a set of nonempty disjoint sets $\Longrightarrow Par = E\circ E_+$.

So,

(i) $Par(x) = E(E_+(x)) = e^{E_+(x)} = e^{e^x - 1}$ (since $E_+(x) = \sum \frac{x^n}{n!} \underbrace{-1}_{\text{no empty sets}}$).

(ii) $\widetilde{Par}(x) = (\widetilde{E\circ E_+})(x) = Z_E(\tilde{E}_+(x), \tilde{E}_+(x^2), \dots) = \frac{1}{(1-\tilde{E}_+(x))(1-\tilde{E}_+(x^2))\cdots} = \frac{1}{(1-x)(1-x^2)\cdots}$.

(iii) $Z_{Par}(x_1, x_2, \dots) = e^{\sum_k \frac{1}{k}\left(e^{\sum_i \frac{x_{ik}}{i}} - 1\right)}$

Species of Permutations.

Every permutation is a set of disjoint cycles $\Longrightarrow S = E\circ C$.

Then, $S(x) = \frac{1}{1-x} = e^{C(x)} \Longrightarrow C(x) = \log\left(\frac{1}{1-x}\right)$.

Another example is that a graph may be identified with a set of connected graphs $\Longrightarrow G = E\circ G^c$.

In general, if two species $F$, $F^c$ are related by $F = E\circ F^c$, then we say that $F^c$ is the species of connected $F$-structures and

(i) $F(x) = e^{F^c(x)}$

(ii) $\tilde{F}(x) = e^{\sum_{k\ge 1} \frac{1}{k} \tilde{F}^c(x^k)}$

(iii) $Z_F(x_1, x_2, \dots) = e^{\sum \frac{1}{k} Z_{F^c}(x_k, x_{2k}, \dots)}$.

Derivatives of Species.

Motivation.

Recall that we want to define some kind of species derivative $G(x) = \frac{d}{dx}F(x)$ based on $|G[n]| = |F[n+1]|$.

So, the number of $G$-structures on $U$ should equal the number of $F$-structures on $U$ + a new element. Call this element $\infty$.

Definition.

An $F'$-structure on $U$ is an $F$-structure on $U^+ = U \sqcup \{\infty\}$, where $\infty = \infty_U$ is an element chosen outside of $U$. For any finite set $U$, we set

$F'[U] = F[U^+]$, $U^+ = U \sqcup \{\infty\}$.

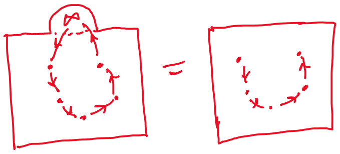
$\sigma^+(u) = \sigma(u)$ if $u \in U$

$F'[\sigma](s) = F[\sigma^+](s)$ for $\sigma : U \to V$ bijection, $\quad \infty$ if $u = \infty$.

## Example.

$C$ is the species of cyclic permutations. A $C'$-structure on $U = \{a, b, c, d, e\}$ is a $C$-structure on $U \cup \{\infty\}$, identified in a natural way with a linear ordering placed on $U$.

Thus, $C' = L \Rightarrow C'(x) = L(x) = \frac{1}{1-x} \Rightarrow C(x) = \int_0^x \frac{dx}{1-x} = \log\left(\frac{1}{1-x}\right)$, as we saw before.



For $F$ a species of structures,

(i) $F'(x) = \frac{d}{dx} F(x)$

(ii) $\tilde{F}'(x) = \left(\frac{\partial}{\partial x_1} Z_F\right)(x, x^2, \dots)$

(iii) $Z_{F'}(\vec{x}) = \left(\frac{\partial}{\partial x_1} Z_F\right)(\vec{x})$.

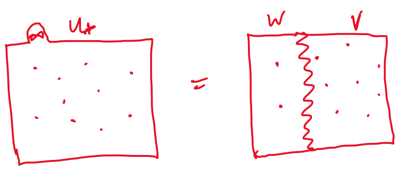
## Example (Partitions).

Par'-structure on $U$ can be identified with a partial partition on $U$, i.e. a partition on some part $V$ of $U$, where $V = U - W$ and $W$ is the class containing $\infty$.

Let $Par_p$ be the species of partial partitions. Then, $Par_p = Par'$.

(i) $Par_p(x) = Par'(x) = \frac{d}{dx} Par(x) = \frac{d}{dx}\left(e^{e^x - 1}\right) = e^{x + e^x - 1}$.

Note: $Par_p = E \cdot Par \Rightarrow Par_p(x) = E(x)\, Par(x) = e^x e^{e^x - 1}$ ✓.




## Remark.

Species differentiation satisfies the usual algebraic properties from Calculus:

- $(F+G)' = F' + G'$
- $(F \cdot G)' = F' \cdot G + F \cdot G'$.
- $(F \circ G)' = (F' \circ G) \cdot G'$.

Done!