Addition and Multiplication of Species

Bergeron — Chapter 1.3

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Week 2

Undergraduate Seminars I

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1·3·1 – Introduction

Given that we now know about species, we must consider the fact that we can perform several operations on them. This is known as combinatorial algebra.

1. Allows us to construct ~~constact~~ construct alternate species
2. Allows us to analyse alternate species
3. Allows us to create associated series

We know that, given two power series of exponential type:

$$\underbrace{f(x) = \sum_{n=0}^{\infty} f_n \frac{x^n}{n!} \qquad g(x) = \sum_{n=0}^{\infty} g_n \frac{x^n}{n!}}$$

$$h(x) = \sum_{n=0}^{\infty} h_n \frac{h^n}{n!}$$ is the general coefficient, h_n

This is constructed from f and g as follows...

$$h = f + g \longrightarrow h_n = f_n + g_n$$

$$h = f \cdot g \longrightarrow h_n = \sum_{i+j=n}^{\infty} \frac{n!}{i!\,j!} f_i g_i$$

$$h = f' \longrightarrow h_n = f_{n+1}$$

take output of g(x) and put into h(x)

$$h = f \circ g \longrightarrow h_n = \sum_{\substack{0 \le k \le n \\ n_1 + \dots + n_k = n}} \frac{n!}{k!\,n_1! \dots n_k!} f_k g_{n_1} \cdots g_{n_k}$$

we know these – they have been derived prior as they consider two species of the same structure

Now, however, we must consider constructions with two seperate species, F and G, in order to create corresponding generating series as such...

$$F + G \longrightarrow (F+G)(x) = F(x) + G(x)$$

$$F \cdot G \longrightarrow (F \cdot G)(x) = F(x) \cdot G(x)$$

$$F \circ G \longrightarrow (F \circ G)(x) = F(G(x))$$

$$F' \longrightarrow F'(x) = \frac{d}{dx} F(x)$$

Note: These identities are created in such a way that displaying their structures are only dependant on displaying F and G's structures individually

Thus, this must be done with the following formulae, replicating that of the exponential ones seen earlier...

$$\text{①}\quad |(F+G)[n]| \longrightarrow |F[n]| + |G[n]|$$

$$\text{②}\quad |(F \cdot G)[n]| \longrightarrow \sum_{i+j=n}^{\infty} \frac{n!}{i!\,j!} |F[i]|\,|G[j]|$$

$$\text{③}\quad |(F \circ G)[n]| \longrightarrow \sum_{j=0}^{n} \sum_{\substack{n_1+n_2+\cdots+n_j=n \\ n_i>0}} \frac{1}{j!} \binom{n}{n_1, n_2 \cdots n_j} |F[j]| \prod_{i=1}^{j} |G[n_i]|$$

$$\text{④}\quad |F'[n]| \longrightarrow |F[n+1]|$$

There could exist many candidates for these definitions, but there are very natural solutions compatible.

> These notes will focus on ① and ②, being addition/subtraction and multiplication/division

1.3.2 - Sum of Species of Structures

As an example, let us consider two species:

1. G^c = species of simple, connected graphs
2. G^d = species of disconnected graphs

We know that a graph must either be connected or disconnected. Therefore, for any finite set U:

$$G[U] = G^c[U] + G^d[U]$$

and thus...

$$G = G^c + G^U$$ → prototype for general definition of adding species

Definition 1: The sum of two species, $F + G$

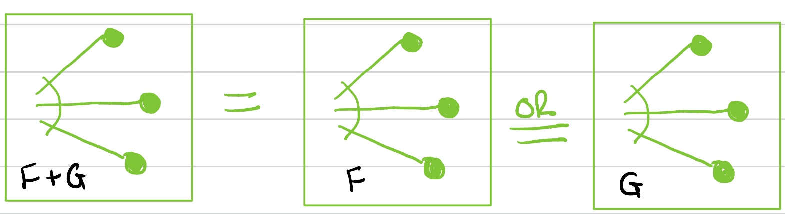
For any finite set U:

$$(F+G)[U] = F[U] + G[U]$$

And trans act along a bijection of $\sigma : U \to V$ on $F+G$ is

$$(F+G)[\sigma](s) = \begin{cases} F[\sigma](s) & \text{if } s \in F[U] \\ G[\sigma](s) & \text{if } s \in G[U] \end{cases}$$

i.e



It is important to note that, where F structures also exist in G (i.e $F[U] \cap F[G] \neq 0$) one must form distinct copies of $F[U]$ and $G[U]$ first...

- Addition is associative (way factors are grouped) and commutative (doesn't matter order), and the empty species 0 is a neutral element...

i.e $F + 0 = 0 + F = F$

To show this, let us consider 3 statements and examples:

a) $(F+G)(x) = F(x) + G(x)$ $\longrightarrow$ $e^x = \cosh(x) + \sinh(x)$

Type Generating series

b) $\widetilde{(F+G)}(x) = \widetilde{F}(x) + \widetilde{G}(x)$ $\longrightarrow$ $\frac{1}{1-x} = \frac{1}{1-x^2} + \frac{x}{1-x^2}$

tilde

c) $Z_{F+G} = Z_F + Z_G$ $\longrightarrow$ $\exp\left(x_1 + \frac{x_2}{2} + \frac{x_3}{3} + \dots\right) = e^{\left(\frac{x_2}{2} + \frac{x_4}{4} + \dots\right)}\left(\cosh\left(x_1 + \frac{x^3}{3} + \dots\right)\right) + \sinh\left(x_1 + \frac{x_3}{3} + \dots\right)$

Considering $E = E_{even} + E_{odd}$ where: E_{even} is set containing an even number; E_{odd} is set containing an odd number

$\boxed{\text{even} \rightarrow f(x) = f(-x)}$

Definition 2: The sum of a family of species $(F_i)_{i \in I}$

A family of species is summable for any finite set, U, if $F_i[U] = \emptyset$, except for a finite number of indicies $i \in I$

This is defined as follows, where $\sigma : U \rightarrow V$ is a bijection and $(s,i) \in \left(\sum_{i \in I} F_i\right)[U]$

a) $$\left(\sum_{i=I} F_i\right)[U] = \sum_{i=I} F_i[U] = \bigcup_{i \in I} F_i[U] \times \{i\}$$

b) $$\left(\sum_{i=I} F_i\right)[\sigma](s,i) = (F[\sigma](s), i)$$

In the same way as previously, let us show this with 3 examples:

a) $\left(\sum_{i \in I} F_i\right)(x) = \sum_{i \in I} F_i(x)$

b) $\left(\widetilde{\sum_{i \in I} F_i}\right)(x) = \sum_{i \in I} \widetilde{F_i}(x)$

c) $Z_{(\sum_{i \in I} F_i)} = \sum_{i \in I} Z_{F_i}$

Using Canonical decomposition, that being each species gives rise to a finite, countable family of species which sum to the original, we can reflect the examples with the identities:

a) $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \dots + \frac{x^n}{n!} + \dots$

b) $\frac{1}{1-x} = 1 + x + x^2 + \dots + x^n + \dots$

c) $\exp\left(x_1 + \frac{x_2}{2} + \dots\right) = \sum_{n \geq 0} \sum_{k_1 + 2k_2 + 3k_3 + \dots = n} \frac{x_1^{k_1} x_2^{k_2} x_3^{k_3} \dots}{1^{k_1} k_1! \, 2^{k_2} k_2! \, 3^{k_3} k_3! \dots}$

Canonical decomposition

$$F_n[U] = \begin{cases} F[U], & \text{if } |U| = n \\ \emptyset, & \text{otherwise} \end{cases}$$

where F_n is restricted to cardinality n

Now, looking at some actual examples...

The finite sum $F + F + \dots + F$, with n copies of F can be denoted by nF. This clearly follows the same rule set:

a) $(nF)(x) = nF(x)$

b) $(\widetilde{nF})(x) = n\tilde{F}(x)$

c) $Z_{nF} = nZ_F$

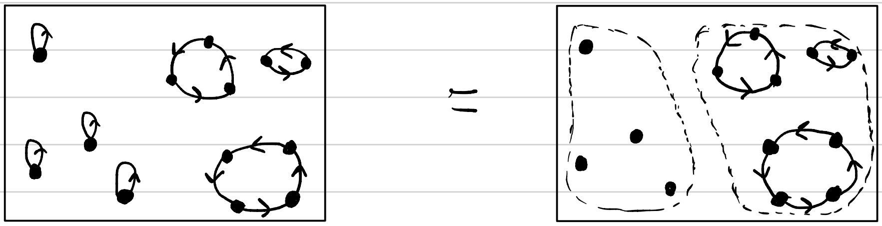
In the instance where $F = 1$...

$$n = \underbrace{1 + 1 + 1 + \dots + 1}_{n} = n \cdot 1$$

And on the set $U = 0$, has no structure.

1·3·3 – Product of Species of Structures

Let us consider the example of the following permutation :



A set of fixed points

A derangement of remaining elements (not in a loop)

Within this, any permutation is clearly comparable with one another

Thus, the species of permutations, S, is the product of the species, E, of sets with the species Der of derangements, forming:

$$S = E \cdot Der$$

no element in its original position

This is a typical example of where multiplication of species is necessary, thus we must define it :

<u>Definition 3</u>: The product of two species, $F \cdot G$, for any finite set, U, can be denoted as $\sum_{(U_1, U_2)} F[U_1] \times G[U_2]$

This definition comes into rise when we define characteristics of an $(F \cdot G)$ series on the set U :
The result is an ordered pair, $s = (f, g)$ where...

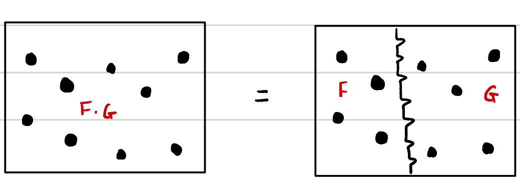
a) f is a F-structure (from F species) on a subset $U_1 \subseteq U$ (inside of)
b) g is a G-structure (from G series) on a subset $U_2 \subseteq U$
c) (U_1, U_2) make up U, where $U_1 \cup U_2 = U$ and $U_1 \cap U_2 = \emptyset$

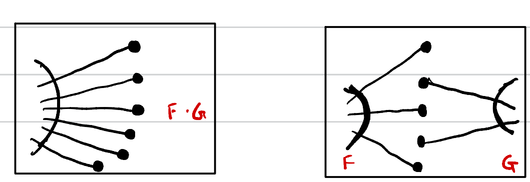
And transport along a bijection of $\sigma : U \rightarrow V$ on $F \cdot G$ for each $s = (f, g)$ is :

$$(F \cdot G)[\sigma](s) = (F[\sigma_1](f), G[\sigma_2](g))$$

with σ representing U_1 and U_2 split

This can be represented more easily by diagrams...





Multiplication Rules

- Associative and commutive up to isomorphism
- 1 is the neutral element, 0 is the absorbing one
  ↳ $1 \cdot F = F \cdot 1 = F$ ↳ $F \cdot 0 = 0 \cdot F = 0$
- It is distributive i.e. $F \cdot (F + G)$

Let us consider three equalities based on this definition...

a) $(F \cdot G)(x) = F(x)\,G(x)$

b) $\widetilde{(F \cdot G)}(x) = \tilde{F}(x)\,\tilde{G}(x)$

c) $Z_{F \cdot G}(x_1, x_2, x_3, \ldots) = Z_F(x_1, x_2, x_3, \ldots)\, Z_G(x_1, x_2, x_3, \ldots)$

These must hold true for the following combinatorial equations based on the example prior...

a) $\frac{1}{1-x} = e^x \, Der(x)$

b) $\prod_{k \geq 1} \frac{1}{1-x^k} = \frac{1}{1-x} \widetilde{Der}(x)$

c) $\prod_{k \geq 1} \frac{1}{1-x_k} = \exp\left(x_1 + \frac{x_2}{2} + \frac{x_3}{3} + \ldots\right) Z_{Der}(x_1, x_2, x_3 \ldots)$

We can rearrange these to get expressions for series associated with the Derangement species:

a) $Der(x) = \frac{e^{-x}}{1-x}$

b) $\widetilde{Der}(x) = \prod_{k \geq 2} \frac{1}{1-x^k}$

c) $Z_{Der}(x_1, x_2, \ldots) = e^{-(x_1 + \frac{x_2}{2} + \frac{x_3}{3} + \ldots)} \prod_{k \geq 1} \frac{1}{1-x^k}$

This links to the classical formula...

$d_n = n!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \ldots + \frac{(-1)^n}{n!}\right)$, which can be used to figure out numbers in fix Der $[n_1, n_2 \ldots]$

If we once again consider the finite species $F + F + \dots + F$, denoted as nF, is also the equivalent of the product between n and F species...

$$nF = n \cdot F$$

Thus, further proof for identifying n with a species.

* These multiplication rules can be used for other combinatorial definitions...

Recalling the species of subsets of a set, $\wp[U] = \{S \mid S \subseteq U\}$, we can achieve the combinatorial equality

$$\wp = E \cdot E \quad , \text{ where } E = e^x$$

$$\therefore \quad \wp(x) = e^x e^x = e^{2x}$$

$$\tilde{\wp}(x) = \frac{1}{(1-x)^2}$$

Thus, using the following series for the index series $Z_\wp$, we obtain immediately:

$$Z_\wp(x_1, x_2, x_3 \dots) = \left(\exp\left(x_1 + \frac{x_2}{2} + \frac{x_3}{3} + \dots\right)\right)^2$$

And deduce the expression:

$$\operatorname{fix} \wp[n_1, n_2, \dots] = 2^{n_1 + n_2 + \dots}$$

The species $\wp^{[k]}$, denoting subsets of $\underset{\substack{\text{number of elements in a subset}}}{\text{cardinality}}$ satisfies the combinatorial equality...

$$\wp^{[k]} = E_k \cdot E \quad , \text{ where } E_k \text{ denotes the species of sets of cardinality } k \text{ (k elements)}$$

Using this, we can derive the following equalities:

a) $\wp[k](x) = e^x \dfrac{k^x}{k!}$ ⟵ $\boxed{\text{COMBINATORIAL INTERPRETATION OF BINOMIAL COEFFICIENTS}}$ $\quad |\wp^{[k]}[n]| = \binom{n}{k}$ ⟶ PTO...

b) $\wp^{[k]}(x) = \dfrac{x^k}{1-x}$,

c) $Z_{\wp^{[k]}}(x_1, x_2, x_3, \dots) = \exp\left(x_1 + \frac{x_2}{2} + \dots\right) \times \sum\limits_{n_1 + 2n_2 + \dots = k} \dfrac{x_1^{n_1} x_2^{n_2} x_3^{n_3}}{1^{n_1} n_1! \, 2^{n_2} n_2! \dots}$

Using this binomial equality and the following combinatorial equality

$$\sum_{k\geq 0} \wp^{[k]} = \wp = E^2$$

We can derive the identity:

$$\sum_{k\geq 0} \binom{n}{k} = 2^n$$

Definition 2: Through associativity, we can apply multiplication rules to finite families F_i of species

Where the product of $\underbrace{F \cdot F \cdots F}_{k}$ is denoted F^k

To do this, the product must be defined as follows...

$$(F_1 \cdot F_2 \cdot F_3)[U] = \sum_{U_1 + U_2 + \cdots = U} F_1[U_1] \times F_2[U_2] \times \cdots \times F_k[U_k]$$

This is essentially doing the same as simple multiplication, with the sum taken over all families (U_i) between $i=1$ and $i=k$ → The union of these is U.

The transport along a bijection, $\sigma : U \to V$ is defined in the following component breakdown

$$(F_1 \cdot F_2 \cdots F_k)[\sigma]((s_i)_{1\leq k\leq i}) = (F_i[\sigma_i](s_i))_{1\leq i\leq k}$$

For $s_i \in F_i[U_i], i = 1, \dots, k$

Definition 3: The product of an infinite family of species can be defined given the family is multipliable

e.g. Consider the species L of linear orderings and its restriction L_k, to sets length k

a) $L_k = X^k \quad k = 0, 1, 2, \dots$

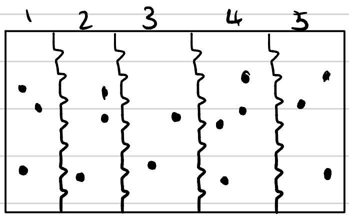
b) $L = 1 + XL = \sum_{k\geq 0} X^k = \prod_{i\geq 0} (1 + X^{2^i})$

Summation Example

Taking the species of non-empty sets, $F = E_+$, we obtain

$Bal^{[k]} = (E_+)^k$, which is summable

A ballot structure, having k levels, looks as such...



Using this, we can define the following

a) $Bal^{[k]}(x) = (e^x - 1)^k$

b) $\widetilde{Bal}^{[k]}(x) = \left(\frac{x}{1-x}\right)^k$

c) $Z_{Bal^{[k]}}(x_1, x_2 \dots) = \left(\exp\left(x_1 + \frac{x^2}{2} + \frac{x_3}{3} + \dots\right) - 1\right)^k$

Since this is summable:

$$Bal = \sum_{k \geq 0} Bal^{[k]} = \sum_{k \geq 0} (E_+)^k$$

This summation thus can be represented in the 3 cases as...

a) $Bal(x) = \frac{1}{2 - e^x}$

b) $Bal(x) = \frac{1-x}{1-2x}$

c) $Z_{Bal}(x_1, x_2 \dots) = \frac{1}{2 - \exp\left(x_1 + \frac{x_2}{2} + \frac{x_3}{3} + \dots\right)}$