2.3 The Calculus of Formal Exponential Generating Functions

In this section we will investigate the analogues of the rules in the preceding section, which applied to *ordinary* power series, in the case of *exponential* generating functions.

Definition 2.13 The symbol $f \overset{egf}{\longleftrightarrow} \{a_n\}_0^\infty$ means that the series f is the exponential generating function of the sequence $\{a_n\}_0^\infty$, i.e., that

$$f = \sum_{n\geq 0} \frac{a_n}{n!} x^n.$$

Let's ask the same questions as in the previous section by supposing that $f \overset{egf}{\longleftrightarrow} \{a_n\}_0^\infty$. Then what is the egf of the sequence $\{a_{n+1}\}_0^\infty$? We claim that the answer is f', because

$$\begin{aligned} f' &= \sum_{n=1}^{\infty} \frac{n a_n x^{n-1}}{n!} \\ &= \sum_{n=1}^{\infty} \frac{a_n x^{n-1}}{(n-1)!} \\ &= \sum_{n=0}^{\infty} \frac{a_{n+1} x^n}{n!}, \end{aligned}$$

which is exactly equivalent to the assertion that $f' \overset{egf}{\longleftrightarrow} \{a_{n+1}\}_0^\infty$.

Hence the situation with exponential generating functions is just a trifle simpler, in this respect, than the corresponding situation for ordinary power series. Displacement of the subscript by 1 unit in a sequence is equivalent to action of the operator D on the generating function, as opposed to the operator $(f(x) - f(0))/x$, in the case of opsgf's. Therefore we have, by induction:

Rule 1′ *If $f \overset{egf}{\longleftrightarrow} \{a_n\}_0^\infty$ then, for integer $h \geq 0$,*

$$\{a_{n+h}\}_0^\infty \overset{egf}{\longleftrightarrow} D^h f. \tag{2.13}$$

The reader is invited to compare this Rule 1′ with Rule 1 on page 36.

Example 2.14 To get a hint of the strength of this point of view in problem solving, let's find the egf of the Fibonacci numbers. Now, with just a glance at the recurrence

$$F_{n+2} = F_{n+1} + F_n \qquad (n \geq 0),$$

we see from Rule 1′ that the egf satisfies the differential equation

$$f'' = f' + f.$$

At the corresponding stage in the solution for the ops version of this problem, we had an equation to solve for f that did not involve any derivatives. We solved it and then had to deal with a partial fraction expansion in order to find an exact formula for the Fibonacci numbers. In this version, we solve the differential equation, getting

$$f(x) = c_1 e^{r_+ x} + c_2 e^{r_- x} \qquad (r_\pm = (1 \pm \sqrt{5})/2)$$

where c_1 and c_2 are to be determined by the initial conditions (which haven't been used yet!) $f(0) = 0$; $f'(0) = 1$. After applying these two conditions, we find that $c_1 = 1/\sqrt{5}$ and $c_2 = -1/\sqrt{5}$, from which the egf of the Fibonacci sequence is

$e^{r_+ x}$

$$f = \left(e^{r_+ x} - e^{r_- x}\right)/\sqrt{5}. \tag{2.14}$$

Now it's easier to get the exact formula, because no partial fraction expansion is necessary. Just apply the operator $[x^n/n!]$ to both sides of (2.14) and the formula (1.11) materializes.

To compare, then, the ops method in this case involves an easier functional equation to solve for the generating function: it's algebraic instead of differential. The egf method involves an easier trip from there to the exact formula, because the partial fraction expansion is unnecessary. Both methods work, which is, after all, the primary *desideratum*.

$$F_n = \frac{1}{\sqrt{5}}\left(r_+^n - r_-^n\right)$$

$$n = 0, 1, 2 \ldots$$

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2.1.4 $f(x) = c_1 e^{r^+x} + c_2 e^{r^-x}$

$f'(x) = r^+ c_1 e^{r^+x} + r^- c_2 e^{r^-x} \rightarrow c_1 = -c_2$

$f(0) = (c_1 + c_2) = 0$

$f'(0) = r^+ c_1 - r^- c_1 = 1$

$c_1 (r^+ - r^-) = 1$

$c_1 = \dfrac{1}{r^+ - r^-}$

$$\boxed{\begin{array}{l} f = \dfrac{\left(e^{r_+ x} - e^{r_- x}\right)}{\sqrt{5}} \\ F_n = \dfrac{1}{\sqrt{5}}\left(r_{+1}^n - r_1^n\right) \end{array}}$$

$\dfrac{(1+\sqrt{5}) - (1-\sqrt{5})}{2} = 2\sqrt{5}$

$= \sqrt{5}$

$c_1 = \dfrac{1}{\sqrt{5}}$

$c_2 = -c_1$

Rule 2′ *If* $f \overset{egf}{\longleftrightarrow} \{a_n\}_0^\infty$, *and* P *is a given polynomial, then*

$$P(xD)f \overset{egf}{\longleftrightarrow} \{P(n)a_n\}_{n\geq 0}.$$

Next let's think about the analogue of Rule 3, i.e., about what happens to sequences when their egf's are multiplied together. Precisely, suppose $f \overset{egf}{\longleftrightarrow} \{a_n\}_0^\infty$ and $g \overset{egf}{\longleftrightarrow} \{b_n\}_0^\infty$. The question is, of what sequence is fg the egf?

This turns out to have a pretty, and uncommonly useful, answer. To find it, we carry out the multiplication fg and try to identify the coefficient of $x^n/n!$. We obtain

$$\begin{aligned} fg &= \left\{\sum_{r=0}^{\infty} \frac{a_r x^r}{r!}\right\}\left\{\sum_{s=0}^{\infty} \frac{b_s x^s}{s!}\right\} \\ &= \sum_{r,s\geq 0} \frac{a_r b_s}{r!s!} x^{r+s} \\ &= \sum_{n\geq 0} x^n \left\{\sum_{r+s=n} \frac{a_r b_s}{r!s!}\right\}. \end{aligned}$$

$$\frac{n!}{r!(n-r)!}$$

The coefficient of $x^n/n!$ is evidently

$$\begin{aligned} \left[\frac{x^n}{n!}\right](fg) &= \sum_{r+s=n} \frac{n! a_r b_s}{r!s!} \\ &= \sum_r \binom{n}{r} a_r b_{n-r}. \end{aligned}$$

We state this result as:

Rule 3′ *If* $f \overset{egf}{\longleftrightarrow} \{a_n\}_0^\infty$ *and* $g \overset{egf}{\longleftrightarrow} \{b_n\}_0^\infty$, *then* fg *generates the sequence*

$$\left\{\sum_r \binom{n}{r} a_r b_{n-r}\right\}_{n=0}^{\infty}. \tag{2.15}$$

Compare to OGF $= \left\{\sum a_r b_{n-r}\right\}_{n\geq 0}$

$3 = 1 \cdot 2 \cdot 1 = 2$ $\qquad$ $2 = 1 \cdot 1 = 1$

Example 2.15 In (1.44) we found the recurrence formula for the Bell numbers, which we may write in the form

$$b(n+1) = \sum_k \binom{n}{k} b(k) \qquad (n \geq 0; b(0) = 1). \tag{2.17}$$

We will now apply the methods of this section to find the egf of the Bell numbers. This will give an independent proof of Theorem 1.5, since (2.17) can be derived directly, as described in Exercise 7 of Chapter 1.

Let B be the required egf. The egf of the left side of (2.17) is, by Rule 1′, B'. If we compare the right side of (2.17) with (2.15) we see that the egf of the sequence on the right of (2.17) is the product of B and the egf of the sequence whose entries are all 1's. This latter egf is evidently e^x, and so we have

$$B' = e^x B$$

as the equation that we must solve in order to find the unknown egf. But obviously the solution is $B = c\exp(e^x)$, and since $B(0) = 1$, we must have $c = e^{-1}$, from which $B(x) = \exp(e^x - 1)$, completing the re-proof of Theorem 1.5.

$B \cdot$ $\qquad$ $e^{e^x - 1}$ $\qquad$ e^x

$C1 = -C_2$ $\qquad$ $e^x \cdot B =$

$$c_1 e^{(1 \pm \sqrt{5})/2} + c_2 e^{1 \pm \sqrt{5}\, 2} = 0$$

$$c1 \cdot \left((1 \pm \sqrt{5})/2\right) \cdot e^{(1 \pm \sqrt{5})/2 - 1} + \quad = 1$$

a

$$\left(1 \pm \sqrt{5}/2\right) c_2\, e^{(1 \pm \sqrt{5})/2 - 1}$$

2.15 $b(n+1) = \sum \binom{n}{k} b(k) \quad n \geq 0 \quad b(0)=1$

Let $B \to \{b(n)\}$

So $B' \to \{b(n+1)\}$

$e^x = \sum_{i=1} \frac{x^n}{n!} \quad e^x \xrightarrow{egf} \{1_i\}_{i=0}^{n}$

$e^x \cdot B = \sum_{k}^{n} \binom{n}{k} b(k) \cdot 1 =$

So $B' = e^x B$

$B = e^{e^x}$ as $B = e^{e^x} e^{x-1}$

$B(0)=1$ so $e^{e^0} \cdot c$

$\Rightarrow e \cdot C = 1 \Rightarrow C = e^{-1}$

So $B = e^{e^x} \cdot e^{-1} = e^{e^x - 1}$ ✓

Example 2.16 In order to highlight the strengths of ordinary vs. exponential generating functions, let's do a problem where the form of the convolution of sequences that occurs suggests the ops form of generating function. We will count the ways of arranging n pairs of parentheses, each pair consisting of a left and a right parenthesis, into a legal string. A legal string of parentheses is one with the property that as we scan the string from left to right we never will have seen more right parentheses than left.

There are exactly 5 legal strings of 3 pairs of parentheses, namely,

$$((())); \; (()()); \; (())(); \; ()()(); \; ()(()). \tag{2.18}$$

Let $f(n)$ be the number of legal strings of n pairs of parentheses ($f(0) = 1$), for $n \geq 0$.

With each legal string we associate a unique nonnegative integer k, as follows: as we scan the string from left to right, certainly after we have seen all n pairs of parentheses, the number of lefts will equal the number of rights. However, these two numbers may be equal even earlier than that. In the last string in (2.18), for instance, after just $k = 1$ pairs have been scanned, we find that all parentheses that have been opened have also been closed. In general, for any legal string, the integer k that we associate with it is the *smallest* positive integer such that the first $2k$ characters of the string do themselves form a legal string. The values of k that are associated with each of the strings in (2.3.6) are 3, 3, 2, 1, 1. We will say that a legal string of $2n$ parentheses is *primitive* if it has $k = n$. The first two strings in (2.18) are primitive.

How many legal strings of $2n$ parentheses will have a given value of k? Let w be such a string. The first $2k$ characters of w are a primitive string, and the last $2n - 2k$ characters of w are an arbitrary legal string. There are exactly $f(n-k)$ ways to choose the last $2n-2k$ characters, but in how many ways can we choose the first $2k$? That is, how many *primitive* strings of length $2k$ are there?

breal

Lemma 2.17 *If $k \geq 1$, $g(k)$ is the number of primitive legal strings, and $f(k)$ is the number of all legal strings of $2k$ parentheses, then*

$$g(k) = f(k-1).$$

Proof: Given any legal string of $k-1$ pairs of parentheses, make a primitive one of length $2k$ by adding an initial left parenthesis and a terminal right parenthesis to it. Conversely, given a primitive string of length $2k$, if its initial left and terminal right parentheses are deleted, what remains is an arbitrary legal string of length $2k-2$. Hence there are as many primitive strings of length $2k$ as there are all legal strings of length $2k-2$, i.e., there are $f(k-1)$ of them. □

Hence the number of legal strings of length $2n$ that have a given value of k is $f(k-1)f(n-k)$. Since every legal string has a unique value of k, it must be that

$$f(n) = \sum_k f(k-1)f(n-k) \qquad (n \neq 0; f(0) = 1) \tag{2.19}$$

with the convention that $f = 0$ at all negative arguments.

The recurrence easily allows us to compute the values $1, 1, 2, 5, 14, \ldots$ Now let's find a generating function for these numbers. The clue as to which kind of generating function is appropriate comes from the form of the recurrence (2.19). The sum on the right is obviously related to the coefficients of the product of two *ordinary* power series generating functions, so that is the species that we will use.

Let $F = \sum_k f(k)x^k$ be the opsgf of $\{f(n)\}_{n\geq 0}$. Then the right side of (2.19) is *almost* the coefficient of x^n in the series F^2. What is it *exactly*? It is the coefficient of x^n in the product of the series F and the series $\sum_k f(k-1)x^k$. How is this latter series related to F? It is just xF. Therefore, if we multiply the right side of (2.19) by x^n and sum over $n \neq 0$, we get xF^2. If we multiply the left side by x^n and sum over $n \neq 0$, we get $F - 1$. Therefore our unknown generating function satisfies the equation

$$F(x) - 1 = xF(x)^2. \tag{2.20}$$

Here we have a new wrinkle. We are accustomed to going from recurrence relations on a sequence to functional equations that have to be solved for generating functions. In previous examples, those functional equations have either been simple linear equations or differential equations. In (2.20) we have a generating function that satisfies a quadratic equation. When we solve it, we get

$$F(x) = \frac{1 \pm \sqrt{1-4x}}{2x}.$$

Which sign do we want? If we choose the '+' then the numerator will approach 2 as $x \to 0$, so the ratio will become infinite at 0. But our generating function takes the value 1 at 0, so that can't be right. If we choose the '−' sign, then a dose of L'Hospital's rule shows that we will indeed have $F(0) = 1$. Hence our generating function is

$$F(x) = \frac{1 - \sqrt{1-4x}}{2x}. \tag{2.21}$$

This is surely one of the most celebrated generating functions in combinatorics. The numbers $f(n)$ are the *Catalan numbers*, and in (2.42) there is an explicit formula for them. For the moment, we declare that this exercise, which was intended to show how the form of a recurrence can guide the choice of generating function, is over.

X $\quad e^{r+x}$ $\quad e$

$e^{(1\pm\sqrt{5})/2-1}$

$\left(C_1 a + C_2 a\right) = 1$

2.16-2.17

Legal String

Primitive $k=n$

$k=$ small # s.t $L \to R$ k = legal

$(\quad) \mid ()()$
k

$f \cdot g = \sum a_r b_{n-r}$

$g(k) \cdot f(n-k)$

$f(k-1) f(n-k)$

$$xF = \sum_k f(k-1) x^k$$

$$F = \sum_k f(k) x^k$$

$$xF \cdot F = \sum_k f(k-1) f(n-k) = xF^2$$

$$\sum_{n \neq 0} f(n) \cdot x^n = F - f(0) \cdot x^0 = F - 1$$

$$F - 1 = xF^2$$

$$F(x) = \frac{1 \pm \sqrt{1-4x}}{2x}$$

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Catalan #s

X

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e^{r+x}

X

$e^{(1\pm\sqrt{5})/2-1}$

$(C_1 a + C_2 a) = 1$

Example 2.26 For fixed n, find

$$\lambda_n = \sum_k (-1)^k \binom{n}{3k}.$$

We could do this one if we knew the function

$$f(x) = \sum_k \binom{n}{3k} x^{3k},$$

because $\lambda_n = f(-1)$. But $f(x)$ picks out every third term from the series $F(x) = (1+x)^n$, and so

$$\begin{aligned} f(x) &= (F(x) + F(\omega_1 x) + F(\omega_2 x))/3 \\ &= \{(1+x)^n + (1+\omega_1 x)^n + (1+\omega_2 x)^n\}/3. \end{aligned}$$

Thus the numbers that we are asked to find are, for $n > 0$,

$$\begin{aligned} \lambda_n = f(-1) &= \{(1-\omega_1)^n + (1-\omega_2)^n\}/3 \\ &= \frac{1}{3}\left\{\left(\frac{3-\sqrt{3}i}{2}\right)^n + \left(\frac{3+\sqrt{3}i}{2}\right)^n\right\} \\ &= 2\cdot 3^{(n/2-1)} \cos\left(\frac{n\pi}{6}\right). \end{aligned} \tag{2.31}$$

The first few values of the $\{\lambda_n\}_{n\geq 0}$ are $1, 1, 1, 0, -3, -9, -18, \ldots$.

To complete this example we want to prove the helpful property (2.28) of the cube roots of unity. But for every $r > 1$, the rth roots of unity do the same sort of thing, namely

$$\frac{1}{r}\sum_{\omega^r=1} \omega^n = \begin{cases} 1 & \text{if } r \backslash n; \\ 0 & \text{otherwise.} \end{cases} \tag{2.32}$$

Indeed, the left side is

$$\frac{1}{r}\sum_{j=0}^{r-1} e^{(2\pi i j n)/r},$$

which is a finite geometric series whose sum is easy to find and is as stated in (2.32). So, with more or less difficulty, it is always possible to select a subset of the terms of a convergent series in which the exponents form an arithmetic progression. See Exercise 25.

Good to memorize

$$\frac{1}{1-x} = \sum_{n\geq 0} x^n \tag{2.33}$$

$$\log\frac{1}{1-x} = \sum_{n\geq 1} \frac{x^n}{n} \tag{2.34}$$

$$e^x = \sum_{n\geq 0} \frac{x^n}{n!} \tag{2.35}$$

$$(1+x)^\alpha = \sum_k \binom{\alpha}{k} x^k \tag{2.38}$$

$$\frac{1}{(1-x)^{k+1}} = \sum_n \binom{n+k}{n} x^n \tag{2.39}$$

$$\frac{1}{\sqrt{1-4x}}\left(\frac{1-\sqrt{1-4x}}{2x}\right)^k = \sum_n \binom{2n+k}{n} x^n \tag{2.47}$$

$$\left(\frac{1-\sqrt{1-4x}}{2x}\right)^k = \sum_{n>0} \frac{k(2n+k-1)!}{n!(n+k)!} x^n \qquad (k \geq 1) \tag{2.48}$$

$$\begin{aligned} \frac{1}{2x}(1-\sqrt{1-4x}) &= \sum_n \frac{1}{n+1}\binom{2n}{n} x^n \\ &= 1 + x + 2x^2 + 5x^3 + 14x^4 + 42x^5 \\ &\quad + 132x^6 + 429x^7 + 1430x^8 \\ &\quad + 4862x^9 + \cdots \end{aligned} \tag{2.42}$$

$$\begin{aligned} \frac{1}{\sqrt{1-4x}} &= \sum_k \binom{2k}{k} x^k \\ &= 1 + 2x + 6x^2 + 20x^3 + 70x^4 + 252x^5 + 924x^6 \\ &\quad + 3432x^7 + 12870x^8 + 48620x^9 + \cdots \end{aligned} \tag{2.43}$$