$x^2 \ln(y) - ty^2 + te^x = e - 1$

$2x \ln(y) - y^2 + e^x + \frac{x^2}{y} - 2xy$

$2x \ln y \cdot \frac{x^2}{y} = y^2 \cdot -2xy + e^x \cdot 0$

Lecture Notes

Intro: Generating function - clothesline hang up series of unknown #'s a_n

↳ $F(x) = \sum_{n \geq 0} a_n x^n$

power series - infinite series of the form of a polynomial w/ infinite # of terms

1.1 two-term recurrence - n^{th} term of a sequence is equal to a given combination of preceding terms

Ex: 1.1 → certain sequence of #'s satisfies $\boxed{a_{n+1} = 2a_n + 1}$, $(n \geq 0, a_0 = 0)$

↳ Find the sequence ↳ (1.1)

$a_0 = 0,\ a_1 = 1,\ a_2 = 3,\ a_3 = 7,\ a_4 = 15,\ a_5 = 31$

↳ look like 1 less than powers of 2.

↳ $a_n = 2^n - 1 \ (n \geq 0)$

What about generating functions method?

↳ instead of finding sequence $\{a_n\}$, find generating function

$\boxed{A(x) = \sum_{n \geq 0} a_n x^n}$

↳ function would tell us explicit formula for the a_n's by expanding $A(x)$ in a series

Find $A(x)$ by multiplying both sides of recurrence relation $(a_{n+1} = 2a_n + 1)$ by (x^n) and sum of values over n which the recurrence is valid $\left(\sum_{n \geq 0}\right)$

↳ gives $\sum_{n \geq 0} a_{n+1} x^n = a_1 + a_2 x + a_3 x^2 + a_4 x^3 + \ldots$

↳ compare to $A(x) \left(= \sum_{n \geq 0} a_n x^n\right)$ → almost same (a_n vs. a_{n+1})

↳ however, rewrite $\sum_{n \geq 0} a_{n+1} x^n = \{(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots) - a_0\}/x$

$= \boxed{A(x)/x \quad \text{since } a_0 = 0}$

↳ left side of 1.1

Now, right side of 1.1 (multiply x^n and sum over all $n \geq 0$)

↳ $\sum_{n\geq 0} (2a_n + 1)x^n = 2A(x) + \sum_{n\geq 0} x^n$

$= \boxed{2A(x) + \frac{1}{1-x}}$ ← geometric series evaluation valid when $|x| < 1$

Now looking at both sides of 1.1

↳ $\boxed{\frac{A(x)}{x} = 2A(x) + \frac{1}{1-x}}$

$A(x) = \frac{x}{(1-x)(1-2x)} \xrightarrow{\text{partial fractions}} x\left(\frac{2}{1-2x} - \frac{1}{1-x}\right)$

$= \{2x + 2^2x^2 + 2^3x^3 + \ldots\} - \{x + x^2 + x^3 + \ldots\}$

$= (2-1)x + (2^2-1)x^2 + (2^3-1)x^3 + \ldots$

↳ coefficient of x^n (a_n) is $(2^n - 1)$, $n \geq 0$

.2 Harder Two term Recurrence (1.2)

A certain sequence satisfies $\boxed{a_{n+1} = 2a_n + n}$ $(n \geq 0,\ a_0 = 1)$

↳ 1, 2, 5, 12, 27, 58, 121...

↳ no obvious general formula, so use GFs.

Look for function $A(x) = \sum_{j\geq 0} a_j x^j$ instead of sequence a_0, a_1

↳ as mentioned earlier, once we find the function, we can identify the sequence as the sequence of power series co-efficients of the function (1)

$(1.2) \Rightarrow a_{n+1} = 2a_n + n$

$A(x) = \{a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots\}$

Steps:

1. make sure recurrence relation that we're trying to solve has clear ranges of values of the subscript for which it is valid
   ↳ (1.2) is valid for $n \geq 0$

2. Define GF that we're looking for
   ↳ looking for sequence $a_0, a_1, a_2 \dots$
   ↳ function $\boxed{A(x) = \sum_{j \geq 0} a_j x^j}$ is a natural choice

3. take recurrence relation (1.2) and multiply both sides by (x^n) and sum over all values of n for which the relation is valid $(\sum_{n \geq 0})$
   ↳ left side of (1.2)

$$a_1 + a_2 x + a_3 x^2 + a_4 x^3 + \dots = (A(x) - a_0)/x$$
$$= \boxed{(A(x) - 1)/x}$$

multiply right side of (1.2) by same thing
↳ $\boxed{2A(x) + \sum_{n \geq 0} n x^n}$

↳ Series: $x + 2x^2 + 3x^3 + 4x^4 + \dots$
↳ need to identify series
work out — look up

$$\sum_{n \geq 0} n x^n = \sum_{n \geq 0} x\left(\frac{d}{dx}\right) x^n = x\left(\frac{d}{dx}\right) \sum_{n \geq 0} x^n = x\left(\frac{d}{dx}\right) \frac{1}{1-x} = \frac{x}{(1-x)^2} \quad (1.3)$$

the series we're interested in is the derivative of the geometric series, so its sum is the derivative of the sum of the geometric series

$A - 1 = 2xA + \frac{x^2}{(1-x)^2}$

$A - 1 - 2xA$

$A(1-2x) \rightarrow \frac{x^2}{(1-x)^2(1-2x)}$

$(1.3) \rightarrow \frac{x}{(1-x)^2}$

$(1.2) \rightarrow a_{n+1} = 2a_n + n$

Results (after multiplying by x^n and $\sum_{n\geq 0}$)

left side $= \frac{(A(x)-1)}{x}$, right side $= 2A(x) + \frac{x}{(1-x)^2}$

set equal, solve for $A(x) \rightarrow \boxed{A(x) = \frac{1-2x+2x^2}{(1-x)^2(1-2x)}}$ (1.5)

Recap

- original problem - find the #'s $\{a_n\}$ that are determined by the recurrence (1.2)
  - ↳ found them: a_n is the coefficient of x^n in power series expansion of the function (1.5)
    - ↳ found GF

To find exact, simple formula for members a_n of the unknown sequence:
- ↳ expand $A(x)$ into a power series
  - ↳ partial fractions easiest when $A(x)$ is a rational function

$$\boxed{\frac{1-2x+2x^2}{(1-x)^2(1-2x)} = \frac{A}{(1-x)^2} + \frac{B}{1-x} + \frac{C}{1-2x}} \quad (1.6)$$

↓ (decomp)

result of the relation: $A(x) = \frac{1-2x+2x^2}{(1-x)^2(1-2x)} = \boxed{\frac{-1}{(1-x)^2} + \frac{2}{1-2x} + \left(\frac{0}{1-x}\right)}$ (1.7)

↳ Find explicit formula for the coefficient of x^n by breaking it down into finding the coefficient of x^n in each of the summands on the right side of (1.7)
- ↳ easier because we know how they expand as a geometric series
- ↳ $\left(\frac{2}{1-2x}\right) \rightarrow$ expands as geometric $\rightarrow 2 \cdot 2^n = 2^{n+1}$ (coefficient of x^n)
- ↳ $\left(\frac{-1}{(1-x)^2}\right) \rightarrow$ as seen in (1.3) $\rightarrow$ coefficient of $x^n = -(n+1)$
- ↳ combine 2 results $\rightarrow$ $\boxed{\text{unknown seq.}: a_n = 2^{n+1} - n - 1}$

Ex: - shows how a single method can replace various specific techniques when dealing w/ sequences

Notation that will help in future

Def. 1.1.- Let $f(x)$ be a series in powers of x. Then by the symbol $[x^n]f(x)$ we will mean the coefficient of x^n in the series $f(x)$.

↳ Ex: $[x^n]e^x = \frac{1}{n!}$, $[t^r]\{1/(1-3t)\} = 3^r$,

$$[u^m](1+u)^s = \binom{s}{m}$$

property of this symbol: $[x^n]\{x^a f(x)\} = [x^{n-a}]f(x)$

another property: let β be any real #, then:

↳ $[\beta x^n]f(x) = (1/\beta)[x^n]f(x)$

Ex: $[x^n/n!]e^x = 1$ for all $n \geq 0$

1.3 A Three Term Recurrence

Fibonacci recurrence → $\boxed{F_{n+1} = F_n + F_{n-1}}$ $(n \geq 1;\ F_0 = 0,\ F_1 = 1)$ (1.10)

Solve for GF: $F(x) = \sum_{n \geq 0} F_n x^n$

↳ multiply both sides of (1.10) by x^n and sum over $n \geq 1$.

left side: $F_2 x + F_3 x^2 + F_4 x^3 + \ldots = \frac{F(x) - x}{x}$

right side: $\{F_1 x + F_2 x + F_3 x + \ldots\} + \{F_0 x + F_1 x^2 + F_2 x^3 + \ldots\} = \{F(x)\} + \{x F(x)\}$

combine: $\frac{F(x) - x}{x} = F(x) + x F(x) \rightarrow \boxed{F(x) = \frac{x}{1 - x - x^2}}$

↳ unknown GF

Find some formulas for Fibonacci #'s by expanding right side into partial fractions
↳ easier when denominator only has first degree factors
↳ $1 - x - x^2 = (1 - x r_+)(1 - x r_-)$
↳ quadratic formula: $r_\pm = \left(\frac{1 \pm \sqrt{5}}{2}\right)$

$$\frac{x}{1 - x - x^2} = \frac{x}{(1 - x r_+)(1 - x r_-)}$$

$$= \frac{1}{(r_+ - r_-)} \left(\frac{1}{1 - x r_+} - \frac{1}{1 - x r_-} \right)$$

thanks to geometric series

$$= \frac{1}{\sqrt{5}} \left\{ \sum_{j \geq 0} r_+^j x^j - \sum_{j \geq 0} r_-^j x^j \right\}$$

pick out coefficient of x^n → $\boxed{F_n = \frac{1}{\sqrt{5}} (r_+^n - r_-^n),\ n = 0, 1, 2, \ldots}$ (1.11)

GF can also give us an approximate answer for when n is large
↳ when n is large $r_+ > 1$ and $|r_-| < 1$
↳ so r_+ will be huge, while r_- will be minuscule

good approx. to F_n: $\boxed{F_n \sim \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n}$ (1.12)

approximation revealing
↳ neglecting 2nd term in (1.11) is a quantity that is tiny, whereas not only is F_n approximately given by (1.12), it is exactly given by integer nearest to right side of (1.12)
↳ approx. found us a simpler, exact formula.

(likely skip)

2.1 Formal Power Series

Proposition 2.1 - A formal power series $f = \sum_{n \geq 0} a_n x^n$ has a reciprocal if and only if $a_0 \neq 0$. In that case the reciprocal is unique

Proof: Let f have a reciprocal, namely, $\frac{1}{f} = \sum_{n \geq 0} b_n x^n$.
Then, $f \cdot \left(\frac{1}{f}\right) = 1$... (Cauchy Product Rule)

"ops" = ordinary power series

2.2 The Calculus of Formal Ordinary Power Series Generating Functions

- Operations on formal series involve corresponding operations on their coefficients
  ↳ think about: which kinds of GF is appropriate for which kind of recurrence relation / other combinatorial situations

Def 2.5 - The symbol $f \overset{ops}{\longleftrightarrow} \{a_n\}_0^\infty$ means that the series f is the ordinary power series ("ops") generating function for the sequence $\{a_n\}_0^\infty$. That is, it means $f = \sum_n a_n x^n$

What generates $\{a_{n+1}\}_0^\infty$?

↳ $\sum_{n\geq 0} a_{n+1} x^n = \frac{1}{x} \sum_{m\geq 1} a_m x^m = \frac{(f(x) - f(0))}{x}$

Therefore: $f \overset{ops}{\longrightarrow} \{a_n\}_0^\infty \Rightarrow \boxed{((f - a_0)/x) \overset{ops}{\longleftrightarrow} \{a_{n+1}\}_0^\infty}$ (2.6)

Shifting - If we shift the subscript by 1 unit, the series represented changes to the difference quotient $(f - a_0)/x$

Ex: shift another unit: $\{a_{n+2}\}_0^\infty \overset{ops}{\longleftrightarrow} \frac{((f - a_0)/x) - a_1}{x} = \frac{f - a_0 - a_1 x}{x^2}$

↳ plug in Fibonacci recurrence relation ($F_{n+2} = F_{n+1} + F_n$, $n \geq 0$, $F_0 = 0, F_1 = 1$)

↳ translates into ops GF $\frac{f - x}{x^2} = \frac{f}{x} + f$

↳ sequence relations → series relations quickly

Rule 1 - If $f \xleftrightarrow{ops} \{a_n\}_0^\infty$, then, for integer $h > 0$,

$$\{a_{n+h}\}_0^\infty \xleftrightarrow{ops} \frac{f - a_0 - \dots - a_{h-1}x^{h-1}}{x^h}$$

Let's multiply sequence by powers of n -

↳ can we express the series $\sum_n n a_n x^n$ in some simple way in terms of the series $f = \sum_n a_n x^n$?

↳ former series is actually exactly xf'

Therefore, to multiply the n^{th} member of a sequence by n causes its ops GF to be "multiplied" by $x(d/dx)$, or xD.

$$f \xleftrightarrow{ops} \{a_n\}_0^\infty \Rightarrow \boxed{(xDf) \xleftrightarrow{ops} \{n a_n\}_0^\infty} \quad (2.7)$$

Ex: Consider the recurrence: $(n+1)a_{n+1} = 3a_n + 1 \quad (n \geq 0, a_0 = 1)$

If f is the ops GF of the sequence $\{a_n\}_0^\infty$, then from rule 1 and (2.7)

↳ $f' = 3f + \frac{1}{1-x}$

Next suppose $f \xleftrightarrow{ops} \{a_n\}_0^\infty$. Then what generates the sequence $\{n^2 a_n\}_0^\infty$?

↳ apply multiply-by-n operator (xD) twice $\rightarrow (xD)^2 f$

In general, $(xD)^k f \xleftrightarrow{ops} \{n^k a_n\}_{n \geq 0}$

Rule 2 - If $f \xleftrightarrow{ops} \{a_n\}_0^\infty$ and P is a polynomial, then

$$P(xD)f \xleftrightarrow{ops} \{P(n)a_n\}_{n\geq 0}$$

Ex. 2.6: Find a closed formula for the sum of the series $\sum_{n\geq 0} (n^2+4n+5)/n!$

applying this rule

$$\{(xD)^2+4(xD)+5\}e^x = \{x^2+x\}e^x + 4xe^x + 5e^x$$

↳ NOT x^2D^2, but operators

$$= (x^2+5x+5)e^x$$

letter x is purely a formal symbol whose powers mark the clotheslines on the line

Rule 3 - If $f \xleftrightarrow{ops} \{a_n\}_0^\infty$ and $g \xleftrightarrow{ops} \{b_n\}_0^\infty$, then

↳ the way 2 ops GF's are multiplied

$$\boxed{fg \xleftrightarrow{ops} \left\{\sum_{r=0}^{n} a_r b_{n-r}\right\}_{n=0}^\infty} \qquad (2.8)$$

3 series → functions f, g, h that generate sequences a, b, and c respectively

↳ $\left\{\sum_{r+s+t=n} a_r b_s c_t\right\}_{n=0}^\infty$

Rule 5 - The effect of dividing an ops GF by $(1-x)$ is to replace the sequence that is generated by the sequence of its partial sums.

if $f \overset{ops}{\longleftrightarrow} \{a_n\}_0^\infty$, then $\frac{f}{1-x} \overset{ops}{\longleftrightarrow} \left\{ \sum_{j=0}^{n} a_j \right\}_{n\geq 0}$

Ex 2.11: Prove that the Fibonacci #'s satisfy: $F_0 + F_1 + F_2 + \dots + F_n = F_{n+2} - 1$, $(n \geq$

using rule 5, the ops GF of the sequence on the left side is $\frac{F}{(1-x)}$, where F is the ops GF of the Fibonacci #'s

in section 1.3, we found F to be $\frac{x}{(1-x-x^2)}$

So by rule 1, the ops GF of the sequence on the right hand side is

$$\frac{F-x}{x^2} - \frac{1}{1-x} \qquad \frac{(F-x)(1-x)-x^2}{x^2(1-x)} \qquad \frac{F-xF-x}{x^2(1-x)}$$

(Ignore) Check: $\frac{\frac{x}{(1-x-x^2)}}{(1-x)} = \frac{\frac{x}{1-x-x^2}-x}{x^2} - \frac{1}{1-x}$ $\qquad \frac{x}{1-x-x^2} = \frac{(F-x)(1-x)}{x^2(1-x)}$ $\qquad \frac{x^3(1-x)}{1-x-x^2} = \frac{x}{1-x-x^2} \; \frac{x^2}{1-x}$

$$\frac{x}{1-x-x^2} = \frac{\left(\frac{x}{1-x-x^2}-x\right)(1-x)}{x^2} - 1 \qquad \frac{1-x^2}{1-x-x^2} = \frac{(F-x)(1-x)}{x^2}$$

(chicken scratch)

$$\frac{(1-x^2)x^2}{1-x-x^2} = \left(\frac{x}{1-x-x^2}-x\right)(\cancel{1-x})$$

$$-\frac{(x+1)\cancel{(x-1)}x^2}{\cancel{1-x}-x^2} = \frac{x}{\cancel{1-x}-x^2} - x \quad \frac{x - x(1-x-x^2)}{\cancel{1-x-x^2}}$$

$$-(x+1)x^2 = x - (x - x^2 - x^3)$$

$$-x^3 - x^2 = -x^2 - x^3 \quad ✓$$