MATH 3951 (UNDERGRADUATE SEMINAR, "TOPICS IN REPRESENTATION THEORY"), INTRODUCTION

The color scheme for this note is a green value of 0.420.

This short note will probably also serve as my notes for the first talk on (probably) Friday. You are not obligated to read through this at all, but I figured I should write this earlier for eager students so they could take a look ahead of time. If you want you can scan (and I really mean scan, at like the rate of two lines a second or faster) through this to convince yourself you chose the right topic for this seminar (species versus category $\mathcal{O}$). Of course not included are things I say regarding organization/logistics.

I understand that I have a tendency to talk a lot when I write, so I'll try to keep these notes short. I'll expound more on the intuition behind things in person.

Also if the presentation in this note seems too fast/advanced, let me know so I can adjust. But also even if things don't make sense, that is what the in-person talk is for – you're not expected to have digested this all (or even any of it) before coming to the talk.

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1. Combinatorial Species

1.1. Prerequisites.

Prerequisites: some calculus, reasonable familiarity with combinatorics.

If you know some category theory that can unify your understanding of this topic, but that is absolutely not required. For reference I learned the basics of this topic in high school, and I did not even know what a basis of a vector space was at the time. Of course knowing some linear algebra will enhance your appreciation of the link between species and representation theory, but again this is not required.

1.2. Vibes/Examples. First you should know what a generating function is. A generating function is just a sum (possibly infinite) of, for example, form

$$F(x) = \sum_{n=0}^{\infty} a_n x^n$$

for whatever numbers a_n. Another example might be

$$F(x) = \sum_{n=0}^{\infty} a_n \frac{x^n}{n!}.$$

These two types of examples are called, respectively, "ordinary generating functions" and "exponential generating functions". These two things are sometimes abbreviated OGF and EGF. This is an infinite sum, so it may not necessarily converge. The analytic information can yield combinatorial knowledge, but to begin we can consider them as formal power series (i.e. ignore convergence problems, and don't plug in any numbers for x).

However, a lot of these times these generating functions can be written in a neater form, like how you can roll a super long string of yarn into a neat yarn ball. For example,

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x$$

is the exponential (remember this from calculus), and

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

is the geometric series from high school.

Briefly let us think about the set of ways you can put n numbers in a circle up to rotation, so that clockwise (123) is the same as clockwise (231). There are $(n-1)!$ ways to do this (convince yourself of this). What if we just chuck this into an exponential generating function, i.e. set $a_n = (n-1)!$ for $\sum_{n=0}^{\infty} a_n x^n/n!$? Then we get

$$C(x) = \sum_{n=0}^{\infty} (n-1)! \frac{x^n}{n!} = \sum_{n=0}^{\infty} \frac{x^n}{n} = \log \frac{1}{1-x},$$

where again you should remember/check this with Taylor series from calculus. (I should maybe disclaim that log defaults to base e in math, not base 10.) So this infinite sum can be written kind of nicely. We have called this C for "cycle".

What about the set of ways you can just put n numbers on the table with no additional structure, i.e. no linear order, no cycles, no nothing? Well there's just one way – throw them on the table. So let's try $a_n = 1$ in the EGF:

$$E(x) = \sum_{n=0}^{\infty} 1 \cdot \frac{x^n}{n!} = e^x.$$

We have called this E for "set" (from French, *ensemble*).

What about the set of permutations on n numbers? Recall the number of permutations is $n!$. For $a_n = n!$ we get

$$P(x) = \sum_{n=0}^{\infty} n! \cdot \frac{x^n}{n!} = \frac{1}{1-x}.$$

We have called this P for "permutation".

Now observe that a permutation is a set of cycles. If we naively translate this from English into math, we might hope to write something like permutation = set(cycle). The magical thing is that it is in fact true that

$$P(x) = E(x) \circ C(x),$$

since

$$\frac{1}{1-x} = e^{\log \frac{1}{1-x}}.$$

This blew my mind when I first saw it, but admittedly this is a rather basic example. Perhaps a more shocking example is that of labeled rooted trees, which I introduced in the first (zeroth?) meeting. Observing that a rooted tree is the root connected to a set of (smaller) rooted trees, we might be tempted to write

$$T(x) = xe^{T(x)},$$

where $T(x)$ is the EGF of the number of rooted trees on n labeled vertices. The shocking thing is this equation is also true, and by using a hammer from analysis called the "Lagrange inversion formula" we can show that the coefficient in front of $x^n/n!$ in $T(x)$ is n^{n-1}, i.e. that there are n^{n-1} rooted trees on n vertices. This fact is very hard to show usually (just look up the proof of Cayley's tree formula), but we have gotten it almost immediately (if you take Lagrange inversion for granted). This should convince you that there is great power to be uncovered here.

Now, what does species aim to do? In short, species is a way of formalizing the bridge between combinatorial observations and algebraic relations of generating functions[1]. So species makes all our vibes above legit.

Then what *is* species? Roughly speaking, a species is a rule associating to an input set, thought of as the label set, an output set, thought of as the set of possible structures of a certain "type" you can build on the input label set. A childish way to say it is that a species is an (infinitely long and big) Lego instructions manual where the n-th page tells you all the different ways you can build a certain type of thing (e.g. a tree or a cycle or a permutation) on n input labeled Lego bricks. Now this Lego manual should be "language-invariant" – it shouldn't matter if you're reading it in Arabic or English or Chinese, and it shouldn't matter if the bricks are labelled in Arabic or English or Chinese, in the sense that the number of structures you can build should be the same. The "functoriality" condition below makes this idea precise.

Definition. A "combinatorial species" $\mathcal{F}$ is a rule/function associating to any input (finite) set U another (finite) set $\mathcal{F}[U]$ such that if $\sigma\colon U \longrightarrow V$ is a bijection of two finite sets, then there is an induced bijection called $\mathcal{F}[\sigma]$ between the finite sets $\mathcal{F}[U]$ and $\mathcal{F}[V]$ satisfying the following two conditions:

(1) If $\sigma\colon U \longrightarrow V$ and $\tau\colon V \longrightarrow W$ are two bijections, then
$$\mathcal{F}[\tau \circ \sigma] = \mathcal{F}[\tau] \circ \mathcal{F}[\sigma]\colon \mathcal{F}[U] \longrightarrow \mathcal{F}[W].$$

(2) For the identity map $\mathrm{id}_U\colon U \longrightarrow U$,
$$\mathcal{F}[\mathrm{id}] = \mathrm{id}_{\mathcal{F}[U]}\colon \mathcal{F}[U] \longrightarrow \mathcal{F}[U]$$
is the identity map on the set $\mathcal{F}[U]$.

These two conditions are called "functoriality", and (in case you know category theory) all the above can be shortened to saying

Definition (category theory definition)**.** A "combinatorial species" is a functor $\mathcal{F}\colon \mathsf{fntSet}^{\times} \longrightarrow \mathsf{fntSet}$ from the category of finite sets with bijections as morphisms to the category of finite sets.

Example. An example of a species might be the rule that associates to an input U the output $\mathtt{Set}[U] := \{U\}$ – this is the "put things on the table with no additional structure" species we discussed earlier. The induced bijections $\mathtt{Set}[\sigma]$ from $\sigma\colon U \longrightarrow V$ are simply $\mathtt{Set}[\sigma] = \mathrm{id}$, and you can check the conditions above easily. $\mathtt{Set}$ is often also denoted E, for ensemble.

Another example might be the species $\mathbf{0}$ which associates to U the empty set
$$\mathbf{0}[U] = \{\}$$
for all U. Exercise: work out the full definition.

Another example might be the species $\mathbf{1}$ which associates to U the set
$$\mathbf{1}[U] = \begin{cases} \{\} & |U| \neq 0 \\ \{\emptyset\} & |U| = 0 \end{cases}.$$
Note that in the second case, the size of the output set is 1. This is a little trippy to think about and I strongly encourage you to work out the full definition, meaning including all the information about $\mathbf{1}[\sigma]$ and all that.

Another example might be the species X which associates to U the set
$$X[U] = \begin{cases} \{\} & |U| \neq 1 \\ \{U\} & |U| = 1 \end{cases}.$$
Note that in the second case the size of the output set is again 1. Again you should work this out in full to test your understanding.

[1] In fancy parlance, species is a categorification of generating functions

Another example might be the species $\texttt{Per}$ which associates to U the set

$$\texttt{Per}[U] = \{\text{set of permutations on the elements of } U\}.$$

This has output size $|U|!$.

A consequence of functoriality is that if your input set U is size n, up to isomorphism of your output set it suffices to consider the case $U = [n] = \{1, \cdots, n\}$. In that case $\mathcal{F}[[n]]$ is shortened to $\mathcal{F}[n]$.

Associated to such a species are three generating functions, or infinite series: the EGF, the OGF, and the cycle index series. I will only define the first one here.

Definition. The EGF of a species, or rather just the generating function of a species $\mathcal{F}$, is

$$F(x) := \sum_{n=0}^{\infty} |\mathcal{F}[n]| \frac{x^n}{n!}.$$

Example. The five species in the last example have, respectively, generating functions

$$e^x, 0, 1, x, \frac{1}{1-x}.$$

You should check this.

These species admit a couple of basic operations, such as addition and multiplication and composition. These things are a little complicated to define.

Definition. Given two species $\mathcal{F}, \mathcal{G}$, addition is defined to be the disjoint union

$$(\mathcal{F} + \mathcal{G})[U] = \mathcal{F}[U] \sqcup \mathcal{G}[U];$$

multiplication is defined to be

$$(\mathcal{F} \cdot \mathcal{G})[U] = \bigsqcup_{U_1 \sqcup U_2 = U} \mathcal{F}[U_1] \times \mathcal{F}[U_2];$$

and composition (provided $\mathcal{G}[\emptyset] = \emptyset$) is defined to be

$$(\mathcal{F} \circ \mathcal{G})[U] = \bigsqcup_{(U_i)_i \text{ partition of } U} \mathcal{F}[\{U_i\}_i] \times \prod_i \mathcal{G}[U_i].$$

The intuition behind this is a little annoying to explain in LaTeX so I'll do it on the blackboard. But in short you should think of addition as "or", multiplication as "and", and composition as "of".

The amazing thing is that these three operations (and more!) translate directly to their series. That is,

Fact. The generating function of $\mathcal{F} + \mathcal{G}$, $\mathcal{F} \cdot \mathcal{G}$, and $\mathcal{F} \circ \mathcal{G}$ are, respectively, $F(x) + G(x)$, $F(x) \cdot G(x)$, and $F(G(x))$.

This is why it is so important that you are all familiar with the calculus of generating functions – the whole point of species is to reduce hard combinatorics problem to the calculus of generating functions, which is supposed to be "easy" (or at least easier).

Now let's do an actual example to show you how powerful this is. Let's count the number of derangements.

Example. Recall a derangement on n numbers is a permutation with no fixed points. How do we count the number of such things? You might know that the principle of inclusion-exclusion can do this for you, but species can do it also.

We begin by making the English observation that, due to the cycle type decomposition of a permutation, a permutation is just a set of fixed points and a derangement. I.e.,

$$\texttt{Per} = \texttt{Set}(X) \cdot \texttt{Der}.$$

Translating to power series, we get

$$P(x) = \frac{1}{1-x} = E(x) \cdot D(x) = e^x \cdot D(x),$$

so that

$$D(x) = \frac{e^{-x}}{1-x}.$$

Now note that $e^{-x} = \sum_{i=0}^{\infty}(-1)^i \frac{x^i}{i!}$ while $\frac{1}{1-x} = \sum_{j=0}^{\infty} x^j$, so that

$$D(x) = \left(\sum_{i=0}^{\infty}(-1)^i \frac{x^i}{i!}\right) \cdot \left(\sum_{j=0}^{\infty} x^j\right) = \sum_{n=0}^{\infty} n! \left(\sum_{i=0}^{n} \frac{(-1)^i}{i!}\right) \frac{x^n}{n!}.$$

Since the coefficient in front of $x^n/n!$ in $D(x)$ is $|\texttt{Der}[n]|$, we conclude that the number of derangements on n is

$$n!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \cdots + (-1)^n \frac{1}{n!}\right).$$

The above example is only a tiny fraction of the full power of this formalism, which we will strive to uncover throughout the semester.

2. Category $\mathcal{O}$

2.1. **Prerequisites.** Prerequisites (for speakers): familiarity with module theory, some homological algebra would be helpful. I also assume you are comfortable with the basic language of representation theory, even if you might not remember the details of the representation theory of finite groups – for example, I expect you to know what an irreducible representation and an indecomposable representation is. People who are not giving talks on category $\mathcal{O}$ will not be expected to know this stuff.

Usually this would have a first course on Lie groups and Lie algebras as a prerequisite, but I will try to abstract this away. Throughout this course you can think of $\mathfrak{sl}_n$ if you don't want to think of general $\mathfrak{g}$, and you can even think of just $\mathfrak{sl}_3$ if you don't want to think of $\mathfrak{sl}_n$. For today let us in particular think of $\mathfrak{sl}_3$.

The nice thing about this is that there are lots of nice pictures you can draw, though maybe they're easiest to draw for $\mathfrak{sl}_3$ – certainly this is the only one I'm familiar with. There is also lots of combinatorics involved, which is either good or bad depending on your taste, but certainly combinatorics is, in general, more accessible than, say, homological algebra. So even for people who don't meet the prerequisites, hopefully the talks can be enjoyable. The pictures are a big part of why I chose this topic over representations of finite-dimensional algebras or even just generic highest weight categories.

Also I should disclaim that we work over $\mathbb{C}$.

2.2. **Vibes.** Here are some things you should know:

Definition. A "Lie algebra" is a vector space $\mathfrak{g}$ equipped with a bracket $[\square, \square]\colon \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}$ such that the bracket is bilinear, antisymmetric, and satisfies the Jacobi identity:

$$[x,[y,z]] + [y,[z,x]] + [z,[x,y]] = 0.$$

The most important example of a Lie algebra for us will be $\mathfrak{sl}_n$, which as a set is

$$\mathfrak{sl}_n = \{x \in \mathrm{Mat}_n : \mathrm{tra}(x) = 0\}$$

the set of traceless $n \times n$ matrices. The bracket on this is defined as

$$[x,y] = xy - yx,$$

where multiplication on the RHS is just regular matrix multiplication.

A representation of a Lie algebra, much like any other representation, is defined

Definition. A "representation of a Lie algebra" is a module (over $\mathbb{C}$ for us) M and a Lie algebra homomorphism

$$\rho\colon \mathfrak{g} \longrightarrow \mathfrak{gl}\, M = \operatorname{End} M,$$

i.e. $\rho([x,y]) = \rho(x)\rho(y) - \rho(y)\rho(x)$.

Note M could well be infinite-dimensional. An important example is the "adjoint representation" $\mathfrak{g}$, upon which $\mathfrak{g}$ acts by

$$\begin{aligned} \operatorname{ad}\colon \mathfrak{g} &\longrightarrow \mathfrak{gl}\,\mathfrak{g} \\ x &\longmapsto [x, \Box] \end{aligned}$$

the bracket.

The Lie algebras whose representations we will care about will admit particularly nice decompositions. They will look roughly like

$$\mathfrak{g} = \left(\bigoplus_{\alpha \in \Phi_+} \mathfrak{g}^{-\alpha} \right) \oplus \mathfrak{h} \oplus \left(\bigoplus_{\alpha \in \Phi_+} \mathfrak{g}^{\alpha} \right),$$

which is called the "triangular decomposition". There are nice pictures you can draw which I won't attempt to replicate on LaTeX. The minus part is called $\mathfrak{n}^-$, and the plus part is called $\mathfrak{n}^+$. This decomposition has lots of properties, which is to say that the structure of this decomposition is rather rigid. Here $\mathfrak{h}$ is a maximal toral subalgebra, which means that it is commutative and consists of elements h such that ad_h is semisimple; Φ_+ is the set of "positive roots" which are distinguished elements of $\mathfrak{h}^*$, see our talk on root systems later; each $\mathfrak{g}^\alpha$ is $\mathfrak{g}^\alpha = \{x \in \mathfrak{g} : [h,x] = \alpha(h)x \ \ \forall\, h \in \mathfrak{h}\}$ and is 1-dimensional; etc., etc.. The space $\mathfrak{h}^*$ has a root system, which adds pretty rich structure to the picture and makes for lots of pretty diagrams. I will draw diagrams in class which are not included here.

The first question you might ask might be what representations of this look like. So it turns out the finite-dimensional representation theory of such a $\mathfrak{g}$ is semisimple, which is either good or bad news depending on your viewpoint. The next question you should ask is what the irreducible representations are. It turns out they are exactly labeled by so-called "dominant integral weights", which are members of $\mathfrak{h}^*$. How might one construct such a thing?

The rest of what I will say is too dependent on pictures... but roughly I will describe, with pictures, what a Verma module is, and how the irreducibles are constructed as the quotient by the maximal submodule. It will not be rigorous – the emphasis is on the pictures.

3. Roadmap

3.1. **Species.** Some references are Wilf's "Generatingfunctionology" and Bergeron et al.'s "Combinatorial Species and Tree-like Structures". We will need to begin the talks with a sequence on the calculus of generating functions. For example, you need to know how to expand $\frac{1-\sqrt{1-4x}}{2x}$. This might look random, but there is certainly a finite (and not very long list) of such things you can memorize before you're able to comfortably formally manipulate these power series. The power series side is supposed to be the easy side of species, after all. After this intro sequence, which will probably follow the early chapters of Wilf, we will begin to follow Bergeron et al..

3.2. **Category $\mathcal{O}$.** There are certainly some basic structural things about Lie algebras that we can't get away with not saying. We will need an intro sequence of talks on root systems, preferably with lots of nice pictures. We will also need a couple of talks explaining the basic features of Lie algebras and the triangular decomposition. These things can reference any intro textbook, e.g. Kirilov's "Introduction to Lie Groups and Lie Algebras". Then we can begin the study of category $\mathcal{O}$, and we'll deal with combinatorics as they come up – this will follow Humphrey's "Representations of Semisimple Lie Algebras in the BGG Category $\mathcal{O}$".

3.3. **Species versus Category $\mathcal{O}$.** There are 12 meetings of this seminar starting next week, i.e. 24 talks. Given the proportions of people interested in species versus category $\mathcal{O}$, maybe let us say we only do 4, at most 5 talks on category $\mathcal{O}$, depending on how people feel. Comparatively, I will be expecting more speed/maturity from the category $\mathcal{O}$ students. I hope this isn't daunting – it's just unfortunate that most people in this class signed up for species, so you will have less time in proportion. So we must move swiftly. First there must be a talk (perhaps, and indeed at most, two) on root systems and Lie algebra basics. Then we will define and study category $\mathcal{O}$, roughly (and I mean extremely roughly – I will probably have to give more input regarding planning) following chapters 1 and 2 of Humphreys across, say, 3 talks.

This means there are at least 19 talks on species. So I'm currently thinking maybe:

- 2 talks (next week!) on the calculus of generating functions, following chapters 1 and 2 of Wilf's book;
- 1 talk on sections 1.0-1.2 in Bergeron, i.e. basics of species;
- 1 talk on section 1.3 in Bergeron, i.e. addition and multiplication of species;
- 1 talk on section 1.4 in Bergeron, i.e. composition and differentiation of species;
- 1 talk on examples of the operations we've talked about so far;
- 1 talk on Mobius inversion somewhere in there;
- 1 talk on section 2.1 in Bergeron, i.e. pointing and the Cartesian product (i.e. the bad multiplication);
- 1 talk on section 2.2 in Bergeron, i.e. functorial composition (scarier name than reality);
- 1 talk on further examples with new tools;
- 1-2 talks on section 2.3 in Bergeron, i.e. weighted species;
- 1-2 talks on section 2.4 in Bergeron, i.e. multivariable species;
- 1-2 talks on section 2.5 in Bergeron, i.e. virtual species;
- 1-2 talks on section 2.6 in Bergeron, i.e. molecular/atomic species.

So you'll note the numbers above add up to at most 18, which is less than 19. I'm sure we can figure something else out for the last talk. But also my schedule above may be a little ambitious, and I don't know how mathematically mature/comfortable you guys are. So this is definitely flexible as we move forward, and we may not even finish covering all the things listed above, which is fine.